Ph.D. Comprehensive Exam
Department of Physics
Georgetown University

Part I: Monday, August 28, 2006, 2:00pm - 6:00pm

Instructions:

- This is a closed-book, closed-notes exam. A calculator may be used for mathematical computations but not for storing formulae.
- Each problem is worth 20 points.
- You should submit work for all of the problems. Note that in some cases, even if you get stuck on one part of a problem, you may be able to make progress on subsequent parts.
- If you need additional paper to work the problems, please use separate sheets for different problems.
- Show all your work.
1. Consider a spin-1/2 particle placed in a magnetic field $\mathbf{B}$ with components:

\[
B_x = 0 \\
B_y = \frac{\sqrt{3}}{2} B_0 \\
B_z = \frac{1}{2} B_0
\]

The Hamiltonian for this system is $H = -\gamma \vec{S} \cdot \vec{B}$, and the Pauli spin matrices are:

\[
S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(1)

in the \{|+\>, |-\>\} basis.

a. Calculate the matrix representing $H$ in the \{|+\>, |-\>\} basis in terms of the frequency $\omega_0 = -\gamma B_0$. 
b. What are the eigenvalues and normalized eigenvectors of $H$ in the $\{|+,\rangle, |-,\rangle\}$ basis.
c. At time $t = 0$, the $z$-component of the spin is measured and found to be $+\hbar/2$. What values can be found if the energy is measured and with what probabilities?
d. Write the state vector $|\psi(t)\rangle$ in the $\{|+, -\rangle\}$ basis.
e. Determine \( \langle S_2 \rangle(t) \).
2. A pair of perfectly conducting parallel wires in the xy-plane forms a set of rails along which a conducting bar of length \( L \) moves with velocity \( v \), as shown in the figure below. Set the origin to be at the back left corner of the structure, labeled \( O \), and let the position of the bar at \( t=0 \) be \( y_0 \). The structure is immersed in a magnetic field, \( B \). Determine the potential, with polarity as indicated, that is induced across the small gap in the loop when:

a) \( B = B_0 \mathbf{k} \)

b) \( B = B_0 \cos(wt) \mathbf{k} \)

c) \( B = B_0 [3 \cos(2\pi x/L) \mathbf{i} - \sin(\pi x/L) \mathbf{k}] \)
3. Consider a three-dimensional free-electron gas of \( N \) electrons occupying a volume \( V \). The energy dispersion is

\[
E_k = \frac{\hbar^2 k^2}{2m},
\]

for plane-wave wave functions

\[
\psi_k(\vec{r}) = \frac{1}{\sqrt{V}} e^{i\vec{k} \cdot \vec{r}}.
\]

(a) Determine the shape of the Fermi surface.

(b) Determine the density of states as a function of energy.
(c) Find an expression for the Fermi wavevector \( k_F \) in terms of the electron density \( n = N/V \).

(d) Show that the average energy per electron at \( T = 0 \) is equal to \( \frac{3}{5} E_F \).
4. Consider a solid alloy of two types of atoms, A and B, with composition \(A_xB_{1-x}\). There are \(N\) lattice sites, with \(n = xN\) of these sites occupied by atoms of type A and \(N - n\) of the sites occupied by atoms of type B.

(a) How many different atomic arrangements are possible for this lattice? Express your answer in terms of \(N\) and \(n\).

(Hint: First suppose the \(N\) atoms are all distinguishable from each other. How many ways are there to arrange \(N\) distinguishable atoms on \(N\) sites? Then adjust this result to take into account the fact that the \(n\) A atoms are indistinguishable from each other, as are the \(N - n\) B atoms.)
(b) Show that the entropy associated with these atomic arrangements (entropy of mixing) is approximately

\[ S = -k_B N[(1 - x) \ln(1 - x) + x \ln x]. \]

You may use Stirling's approximation: \( \ln N! \approx N \ln N - N \) (for \( N \gg 1 \)).
(c) Suppose the nearest-neighbor interatomic interaction energies, are all equal \( u_{AA} = u_{AB} = u_{BB} = u \), and that interactions beyond nearest neighbors are negligible.

i. Write down an expression for the Helmholtz free energy \( F \) of the system, assuming that each lattice site has \( z \) nearest neighbors. (Ignore boundary effects.) Let \( u \) be the interatomic potential energy per nearest neighbor bond.

ii. In equilibrium, would you expect the system to be a homogeneous solid solution in which A and B atoms are arranged randomly on the lattice sites, or would you expect a two-phase arrangement consisting of extended regions of elemental A and extended regions of elemental B? Does the answer depend on the temperature? Explain.
Ph.D. Comprehensive Exam  
Department of Physics  
Georgetown University

Part II: Tuesday, August 29, 2006, 2:00pm - 6:00pm

Instructions:

- This is a closed-book, closed-notes exam. A calculator may be used for mathematical computations but not for storing formulae.
- Each problem is worth 20 points.
- You should submit work for all of the problems. Note that in some cases, even if you get stuck on one part of a problem, you may be able to make progress on subsequent parts.
- If you need additional paper to work the problems, please use separate sheets for different problems.
- Show all your work.
1. The system of thin lenses shown below is used to image the object (arrow) shown on the left. The object is 100. cm in front of the first lens, which has a focal length of 50.0 cm. The two lenses are separated by 150. cm. The focal length of the second lens is -25.0 cm.

![Diagram of two lenses with an object in front]

a. Calculate the location of the image.
b. Find the transverse magnification of the lens system.
c. Trace at least three separate rays from the point of the object to the first image, and three rays from the first to the final image.
2. Consider a two-dimensional crystal lattice containing two species of atoms, A and B. Species A has one valence electron per atom while species B has two valence electrons per atom. Part of the lattice is shown at right. Some fundamental constants and conversion factors that might be useful are provided on the last page of this problem.

(a) In the diagram above, draw a primitive unit cell. How many atoms are there per primitive cell? Write down expressions for a set of primitive lattice vectors, \( \vec{a}_1 \) and \( \vec{a}_2 \), in terms of the nearest neighbor distance \( d \) and the \( \hat{x} \) and \( \hat{y} \) unit vectors.

(b) For a typical solid, roughly what would you expect the interatomic separation \( d \) to be?
(c) A sketch of the lowest four (valence) electron bands along some unspecified direction in reciprocal space is shown below, with \( k_{BZ} \) corresponding to the boundary of the Brillouin zone in that direction.

![Energy vs. k diagram](image)

i. Sketch in the Fermi level \( (E_F) \) in the diagram. Explain how you estimated the location of \( E_F \).

ii. Is this an electrical conductor or insulator? How can you tell?

iii. The zero of energy is set at the bottom of the lowest valence band. Make an order-of-magnitude estimate for \( E_F \) (in eV).
(d) Consider the vibrational modes of this two dimensional lattice. Assume that motion of the atoms is limited to the $x$-$y$ plane.

i. Make a qualitative sketch of the phonon dispersion curves ($\omega$ vs. $k$) below, labeling the acoustic and optic branches. Show how you determined the number branches of each type to include.
ii. Provide a rough (order of magnitude) estimate for the maximum phonon energy (in eV).

iii. In the phonon dispersion plot on the previous page, add a curve to represent the dispersion relation for a photon (keeping the axes at the same scale.) Make sure the photon curve is clearly labeled.

The following information may be useful:
\[ \hbar = 1.05 \times 10^{-34} \text{ J-s}, \quad k_B = 1.38 \times 10^{-23} \text{ J/K}, \quad c = 3.00 \times 10^8 \text{ m/s}, \quad 1 \text{ eV} = 1.60 \times 10^{-19} \text{ J.} \]
3. A particle of mass $m$ is trapped in between two rigid walls in 1D:

$$V(x) = 0 \text{ for } 0 < x < L$$
$$V(x) = \infty \text{ otherwise.}$$

At $t = 0$ the particle is known to be exactly at $x = L/2$

a) What is the momentum of the particle at $t = 0$? Explain.

b) Write down the wavefunction for $t=0$. 
c) Find the relative probabilities for the particle to be found in the various energy eigenstates at \( t=0 \). You may leave your answer in terms of the normalization factors for the energy eigenstates (i.e., don't worry about normalizing the energy eigenstates).
d) Write down the wavefunction for $t > 0$.

e) Phase factors do not change the physical state of a particle. Is the particle in a stationary state? Explain.
4. Consider a three-dimensional, spin zero, boson, particle-like (or quasiparticle) excitation for which the energy versus wave vector relation is given by

\[ \varepsilon_k = \Delta + \frac{\hbar^2}{2m} (k-k_0)^2. \] (1)

Here \( \Delta, m, \) and \( k_0 \) are positive constants with units of energy, mass, and 1/distance, respectively. For very low temperatures, to excellent approximation, the chemical potential of these excitations is very small,

\[ \Delta > k_B T, \] (2)

and

\[ \frac{\hbar^2 k_0^2}{2m} > k_B T, \] (3)

where \( k_B \) is the Boltzmann constant and \( T \) is the absolute temperature.

These excitations are known as rotons and are found in superfluid He-4.

a. **(5 points)** Show that the grand partition function for this system is given by

\[ Z_G = \prod_k \left[ 1 - e^{-\varepsilon_k - \mu}/k_B T \right]^{-1}, \] (4)

where \( \mu \) is the chemical potential.

Your calculation to arrive at Eq.(4) must be clear and explicit.
b. **(5 points)** Assume periodic boundary conditions and show that the grand potential,

\[ \Phi_g = -k_B T \ln(Z_g), \]  

(5)

can be written as

\[ \Phi_g = k_B T \frac{V}{(2\pi)^3} \int d^3k \ln \left[ 1 - e^{-\Delta/k_B T} e^{-\alpha(k - k_0)^2} \right], \]  

(6)

where \( V \) is the volume of the system and

\[ \alpha = \frac{\hbar^2}{2mk_B T}. \]  

(7)

Your calculation to arrive at Eqs. (6) and (7) must be clear and explicit.
c. (4 points) Integrate by parts to obtain

\[ \Phi_G = - \frac{\hbar^2 V}{6\pi^2 m} \int_0^\infty dk \frac{k^3 \left( k - k_0 \right)}{e^{\Delta/kT} e^{\alpha (k - k_0)^2} - 1}. \]  \hspace{1cm} (8)

Show all your work.
d. (3 points) Write the numerator of the integrand in Eq. (8), \(k^3(k-k_0)\), as a fourth order polynomial in \((k-k_0)\). Then make use of the facts that

\[
e^{\frac{\Delta}{k_BT}} > 1
\]  

and

\[
q_0^2 > 1,
\]

where

\[
q_0 = \sqrt{\alpha} k_0,
\]

to show that to good approximation

\[
\Phi_G = -AV \frac{m^p k_0^q}{\hbar} (k_B T)^r e^{-\frac{\Delta}{k_BT}},
\]

where the parameters \(A\), \(p\), \(q\), and \(r\) are positive numerical constants which are to be determined in solving the problem.

**HINT:**

\[
\int_0^\infty dx \, x^2 e^{-x^2} = \frac{\sqrt{\pi}}{4}.
\]
e. **(3 points)** Within the stated approximations, calculate the heat capacity at constant volume and comment on its behavior for T → 0.