INSTRUCTIONS:

(1) Please answer each of the four questions on separate pieces of paper.
(2) Please write only on one side of a sheet of paper
(3) Please write in pen only
(4) When finished, please arrange your answers alphabetically (in the order in which they appeared in the questions, i.e. 1 (a), 1 (b), etc.)
1. There are two goods and one consumer. Let \( x_i \) \((i = 1, 2)\) denote the quantity of good \( i \) consumed by the consumer. Preferences of the consumer over the goods are represented by the Cobb-Douglas utility function:

\[
u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}, \quad \text{where } \alpha \in (0, 1).
\]

Let \( p = (p_1, p_2) \gg 0 \) denote the vector of market prices. Let \( w > 0 \) be the consumer’s wealth.

(a) Carefully specify the consumer’s problem. Derive the Walrasian demand functions of the consumer (using the Lagrange method). Be careful to argue, why you can use the Kuhn-Tucker conditions to characterize the consumer’s demand.

(b) Derive the consumer’s indirect utility function. Check that the indirect utility function satisfies the standard properties. Namely, show that the indirect utility function is continuous, homogeneous of degree zero, increasing in wealth and decreasing in prices, and quasi concave.

(c) Suppose that good 2 is a “healthy” good, and good 1 is an “unhealthy” good. There is a politician, who wants to impose a tax \( \tau > 0 \) per unit of consumption of good 1 in order to reduce its consumption. At the same time, the politician cares about being re-elected for the next term in office, so she wants to give the consumer a lump-sum subsidy \( T \) as a compensation for taxation of the “unhealthy” good. Suppose that the consumer will re-elect the politician if he is at least as well off after the tax-subsidy policy as before it.

Assuming that the politician would not like to overcompensate the consumer, write down the politician’s problem. Find the minimal lump-sum transfer, which will guarantee the politician’s re-election.

\textit{Hint:} You will save some time if, instead of solving the politician’s problem, you use the duality between utility maximization and expenditure minimization problems.

(d) Let us call the tax-subsidy policy sustainable, if the tax revenue (collected from good 1 sales) can at least cover the lump-sum subsidy. Characterize the range of prices and tax rate \( \tau \), for which the policy is sustainable.
2.

(a) A decision-maker (DM) with von Neuman-Morgenstern expected utility is asked to rank three probability distributions of a random variable $X$ specified below by cumulative distribution functions (c.d.f.).

\[
F_1(x) = \begin{cases} 
0 & \text{if } x < 40 \\
\frac{1}{2} & \text{if } 40 \leq x < 80 \\
1 & \text{if } x \geq 80
\end{cases}
\]

\[
F_2(x) = \begin{cases} 
0 & \text{if } x < 30 \\
\frac{3}{5} & \text{if } 30 \leq x < 105 \\
1 & \text{if } x \geq 105
\end{cases}
\]

\[
F_3(x) = \begin{cases} 
0 & \text{if } x < 0 \\
\frac{1}{2} & \text{if } 0 \leq x < 60 \\
1 & \text{if } x \geq 60
\end{cases}
\]

Assume that the DM’s expected utility index $u$ is an increasing and strictly concave function. How would the DM rank the above c.d.f.’s according to stochastic dominance? Explain carefully why.

(b) How will your answers to the question in (a) change if the DM’s expected utility index $u$ is an increasing linear function? Explain carefully why.

(c) Suppose that c.d.f.’s $F_1$ and $F_2$ represent payoffs of risky projects that require a fixed sum investment $I$. Let the project with profits given by c.d.f. $F_1$ (respectively, $F_2$) be type 1 (respectively, type 2) project. There is a risk-neutral entrepreneur, who knows the type of a project, and has no money of his/her own. There is a risk-neutral lender, who agrees to invest the capital $I$ in return for $R$. If profit realization of the project is greater than or equal to $R$, the lender is paid back, and the entrepreneur gets the difference between the profit and $R$. If profit realization of the project is less than $R$, the lender gets the profit, and the entrepreneur gets nothing. Write down the expected net gain for each project type as a function of $R$.

(d) Assume that the entrepreneur is rational and will apply for the loan if and only if the expected net gain is non-negative. What is the maximal repayment to the lender, which makes it rational for an entrepreneur with project of type $j \in \{1, 2\}$ to apply for a loan?

If $R = 60$, how would the entrepreneur rank type 1 and type 2 projects? Is the ranking the same as in (b)? Explain why or why not.

(e) Suppose that the lender does not know the type of the project, but assigns probability $\gamma \in (0, 1)$ to type 2 project, and $1 - \gamma$ to type 1 project. What is the c.d.f. of the random variable $X$ as perceived by the lender?

(f) What is the lender’s expected payoff as a function of $R$? Which $R$ maximizes the lender’s expected payoff?
3. Two friends are at a beach for two days. On each day, each must decide whether or not to swim. With probability $\pi \in (0, 1)$, the water is infested with sharks, and if sharks are present, any swimmer will be attacked with probability $p \in (0, 1)$ on day one, and with certainty on day two. A day’s peaceful swim yields a payoff of 1, being attacked by a shark has a payoff of $-c < 0$, and sitting on the beach yields a payoff of 0. If a swimmer is attacked by sharks on the first day, then it is clear to both friends that the water is infested, and that any swimmer will surely be attacked on day two. However, a swimmer not being attacked on the first day does not allow the friends to conclude that there are no sharks, and an attack may still take place on day two. If no one swims on day one, then nothing is learnt. Each player seeks to maximize the sum of her expected payoffs over the two days. Assume that $\pi c > 1 - \pi$, and $\pi c < 2(1 - \pi)$.

(a) Consider first the case where only one person is present at the beach over the two days. Represent this decision problem as an extensive form. Calculate the respective probabilities $\pi_A$, $\pi_{NA}$ and $\pi_{NS}$ the person attributes on day two to the water being infested if she was attacked on day one, if she swam on day one and was not attacked, and if she did not swim on day one. Use backward induction to solve for an optimal strategy for this person, depending on the value of $p \in (0, 1)$. (Assume that the person does not swim when she is indifferent.)

(b) Consider now the case where both friends are present, and assume that $p = 1$. How many proper subgames does this game have? Solve for a pure strategy subgame perfect equilibrium of this two period game.

(c) Consider again the case where both friends are present and $p = 1$. Solve for a symmetric subgame perfect equilibrium of the two period game, where both players swim with probability $q$ in period one.

Now consider the following prisoners’ dilemma, repeated infinitely often. Represent each of the strategy profiles below as an automaton. In each case, determine for which values of the discount factor $\delta \in (0, 1)$ the strategy profile is a subgame perfect equilibrium in the infinitely repeated game, when payoffs are evaluated according to the average discounted payoff criterion.

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<tr>
<td>C</td>
<td>5.5</td>
<td>0.6</td>
</tr>
<tr>
<td>D</td>
<td>6.0</td>
<td>1.1</td>
</tr>
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(d) Each player always plays C.
(e) Each player always plays D.
(f) Player 1 alternates between C and D (starting with C), while player 2 always plays C; if any player deviates from this scenario, then each plays D thereafter.
4. A monopolist can produce a good in different qualities. The cost of producing a unit of quality \( s \geq 0 \) is \( s^2 \). Consumers buy at most one unit each. A consumer of type \( \theta \geq 0 \) has utility function \( u(s|\theta) = \theta s \) if they consume one unit of quality \( s \), and \( u(s|\theta) = 0 \) if they do not consume. The monopolist decides on the quality (or qualities) it is going to produce and corresponding price(s) \( T \geq 0 \). Consumers observe qualities and prices and decide which quality to buy if at all.

(a) Characterize the first-best solution.

(b) Suppose that the seller cannot observe \( \theta \), and suppose that \( \theta = \theta_L \) with probability \( \beta \) and \( \theta = \theta_H \) with probability \( 1 - \beta \), with \( \theta_H > \theta_L > 0 \). Characterize the second-best solution. Is it efficient? What is the consumers’ informational rent? Give an economic intuition for your results.

(c) Suppose now that \( \theta \) is uniformly distributed on the interval \([0, 1]\). Characterize the second-best optimal quality-price schedule. Give an economic intuition for your results.