Ph.D. Comprehensive Exam
Department of Physics
Georgetown University

Part I: Monday, August 29, 2005, 9:00am - 1:00pm

Instructions:

• This is a closed-book, closed-notes exam. A calculator may be used for mathematical computations but not for storing formulae.

• Each problem is worth 20 points. All of the problems will be graded and tabulated to generate a final score. Therefore, you should submit work for all of the problems.

• For convenience in grading, please use only one side of each sheet of paper and work different problems on separate sheets of paper. At the end of the exam, staple together the sheets for each problem separately.

• To receive full credit, please show all your work.
1. Consider a potential of the form:

\[ V(x) = -\alpha \delta(x), \]  

where \( \alpha \) is a positive constant.

(a) Determine the discontinuity \( \frac{d\phi(0+)}{dx} - \frac{d\phi(0-)}{dx} \) in the derivative \( d\phi/dx \) at \( x = 0 \) by integrating the Schrödinger equation from \( -\epsilon \) to \( \epsilon \) and taking the limit \( \epsilon \to 0 \). (Here, \( \phi(x) \) is the wavefunction.) Your answer should be in terms of \( m, \alpha, \) and \( \phi(0) \), where \( m \) is the particle mass.

(b) For energies \( E < 0 \), bound states are found with normalized wavefunctions of the form:

\[ x \leq 0: \quad \phi(x) = Ae^{-\rho x} + A'e^{\rho x} \]  
\[ x \geq 0: \quad \phi(x) = Be^{-\rho x} + B'e^{\rho x} \]

i. What is the value of \( \rho \) in terms of \( m \) and \( E \)?

ii. Determine \( A, A', B, \) and \( B' \) in terms of \( m \) and \( E \).
2. Consider a two-dimensional semiconductor with a direct band gap $\Delta$ at the Brillouin zone center and isotropic parabolic valence and conduction bands with effective masses of $m_v$ and $m_c$, respectively. Let the zero of energy to be at the top of the valence band.

(a) Write an equation for the valence band energy as a function of the magnitude of the wavevector $k = |\vec{k}|$. Do the same for the conduction band.

(b) For a two-dimensional parabolic band with effective mass $m$, the density of electronic states (taking into account spin) is $\frac{m}{\pi \hbar^2}$. Sketch the density of states (DOS) versus energy for this semiconductor.

(c) Making suitable expansions of the Fermi-Dirac function, derive an expression for the chemical potential as a function of temperature, $T$. Assume that the semiconductor is intrinsic (i.e., no impurities). Also assume that on the scale of $k_B T$, the chemical potential lies far from the band edges.
3. The figure below shows a sketch of the electrical resistivity of a solid as a function of temperature:

(a) Is this a metal, insulator, or semiconductor? Explain how you know.

(b) Describe the physical process primarily responsible for the resistance and explain qualitatively the temperature dependence (i) near 0 K and (ii) near 300 K.

(c) Estimate the mean free path and mean free time at T=0K.

(d) Is this material clean? Explain.

Possibly useful information: \( n = 10^{28} \text{m}^{-3}, v_F = 10^8 \text{ cm/s}, m_e = 1.7 \times 10^{-27} \text{ kg}, \) electron charge \( e = 1.6 \times 10^{-19} \text{ C}, \) \( \sigma = ne^2 \tau / m_e. \)
4. In the elementary kinetic theory of statistical mechanics, one studies the mechanics of one particular particle in a large number and then deduces values for macroscopic thermodynamic variables based on ensemble averaging over particles.

Use elementary kinetic theory and carefully defined mathematical notation of your choice to deduce the following two relationships for an ideal gas of \( n \) particles per unit volume at equilibrium:

(a) The gas pressure, \( p \), is related to the average kinetic energy per particle, \( e \), by

\[
p = 2ne/3.
\]

(b) The gas pressure, \( p \), for a mixture of \( M \) different ideal gases is the sum of the partial pressures of each of the \( M \) different gas components.
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Part II: Tuesday, August 30, 2005, 9:00am – 1:00pm

Instructions:

This is a closed-book, closed-notes exam.

A calculator may be used for mathematical computations but not for storing formulae.

Each problem is worth 20 points. All of the problems will be graded and tabulated to generate a final score. Therefore, you should submit work for all of the problems.

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To receive full credit, please show all your work.
1. a) Two uniform infinite sheets of charge intersect each other at right angles. The sheets carry opposite charge densities of $+\sigma$ and $-\sigma$. Find the magnitude and direction of the electric field everywhere.

\begin{align*}
  & +\sigma \\
  \hline
  \hline
  -\sigma
\end{align*}

b) An infinitely long wire carries a current $I$. The wire is formed into a semicircle with radius $R$ around the point $P$ as shown in the figure. Find an expression for the magnetic flux density (the $B$ field) at the point $P$. 

\begin{center}
\begin{tikzpicture}
\draw[thick,->] (0,0) -- (1,0);
\draw[thick] (1,0) arc (0:180:1);
\filldraw[white] (1,0) arc (0:180:1) -- (1,0) -- cycle;
\filldraw[black] (1,0) arc (0:180:1) -- (1,0) -- cycle;
\node at (1,0) {$P$};
\draw[thick,->] (0,0) -- (0.5,0); \node at (0.5,0) {$I$};
\end{tikzpicture}
\end{center}
2. Consider a particle in a one-dimensional harmonic oscillator potential.

a) If at the time $t = 0$ the state of the particle is an eigenstate of the Hamiltonian corresponding to the energy eigenvalue $E_n = (n + 1/2) \hbar \omega$, $|\psi(0)\rangle = |n\rangle$, what is the expectation value of the operator $X$ at the time $t$, $<X(t)>$?

b) If at the time $t = 0$ the state of the particle is a linear combination of eigenstates of the Hamiltonian, $|\psi(0)\rangle = 1/\sqrt{2} (|n\rangle + |n-1\rangle)$, what is the expectation value of the operator $X$ at the time $t$, $<X(t)>$?

Next consider a particle in a two-dimensional isotropic harmonic oscillator potential, $1/2 k (x^2 + y^2)$, where $k$ is a constant. The state of the particle is a non-degenerate eigenstate of the angular momentum $L_z$.

c) If you measure the energy of the particle, will the state of the particle change? Explain by using the symmetry properties of the system (no calculations required).

Last, consider two non-interacting particles, both of mass $m$, in the same spin state and in a one-dimensional harmonic oscillator potential. The energy of one particle is $(3/2) \hbar \omega$ and the energy of the other is $(5/2) \hbar \omega$.

d) Write the state of the two-particle system in terms of the normalized single-particle states $|1\rangle$ and $|2\rangle$, assuming that the two particles are identical fermions. What is the total energy of the two-particle system?

e) Write the state of the two-particle system assuming that the particles are distinguishable. What is the total energy of the system in this case?

Useful formulas:

$X = (\hbar/2m\omega)^{1/2} (a + a^\dagger)$;

$a |n\rangle = (n)^{1/2} |n - 1\rangle$

$a^\dagger |n\rangle = (n + 1)^{1/2} |n + 1\rangle$
3. Consider a solid which has $N$ sites on which particles can be localized.

Each site can be viewed as a potential well of depth $V_0$ in the following sense: a site state, taking into account all of the particles associated with the site, with energy $> V_0$ decays very quickly and thus is not present when the solid is in equilibrium.

Note that we have, for convenience, chosen the zero of energy at the bottom of the potential well.

Each site has associated with it just two distinct quantum mechanical single-particle states with energies $\varepsilon_1$ and $\varepsilon_2$. If two or more particles are localized on a given site, then there is an additional energy, $\Delta$, arising from each particle-particle interaction. There is no interaction between two particles if they are on different sites.

The four energies in the problem are constrained by the relations

$$0 < \varepsilon_1 < \varepsilon_2,$$  \hspace{1cm} (1)

$$0 < \varepsilon_1 + \varepsilon_2 < V_0,$$  \hspace{1cm} (2)

$$\Delta > 0,$$  \hspace{1cm} (3)

and

$$\frac{V_0}{3} - \varepsilon_1 < \Delta < V_0 - 2 \varepsilon_2.$$  \hspace{1cm} (4)

a. Obtain the grand partition function for one site and from it deduce the grand partition function for a solid with $N$ sites, for the case where the particles are identical fermions.

b. Obtain the grand partition function for one site and from it deduce the grand partition function for a solid with $N$ sites, for the case where the particles are identical bosons.

c. For both cases, work out the average number of particles per site.

HINT: Begin the problem by determining the allowed values of $n_1$ and $n_2$, the occupation numbers for the first and second quantum mechanical single-particle states associated with a site.
4. Dielectrics and Capacitors: (20 points)

The following figure shows three parallel plate capacitors (also called condensers) of the same area, $A$, and plate separation, $d$. Designate $C_0$ to be the capacitance of the vacuum condenser. The other two parallel plate configurations are each modified by half-filling them with a dielectric goo having a dielectric constant of $\varepsilon$. They are filled differently, as shown in the figure below.

![Diagram of capacitors](image)

$C_0$ $\quad$ $C_1 = ?$ $\quad$ $C_2 = ?$

a) Find the capacitance of each of these two goo-filled condensers (ie. $C_1$ and $C_2$) as a function of the original capacitance $C_0$, keeping in mind that,

\[ C_0 = \frac{A}{4\pi d} = \frac{Q_0}{V} \quad \text{and} \quad C_{\text{with dielectric}} = \frac{\varepsilon A}{4\pi d} = \frac{\varepsilon Q_0}{V} \quad \text{(cgs units)} \]

or, if you prefer,

\[ C_0 = \frac{\varepsilon_0 A}{d} = \frac{Q_0}{V} \quad \text{and} \quad C_{\text{with dielectric}} = \frac{\varepsilon A}{d} = \frac{\varepsilon Q_0}{V} \quad \text{(in SI units)} \]

where $\varepsilon_0$ (dielectric constant of free space) = $8.854 \times 10^{-12}$ F/m

and $\varepsilon$ (dielectric constant) = $\varepsilon_0 \varepsilon_r$ (relative permittivity)
b) If the original vacuum condenser had 25 cm x 25 cm metal plates, separated from each other by a distance of 100 μm, what is the value of capacitance $C_0$ in cgs. From this, what diameter sphere, in free space, would have the same capacitance (to the nearest foot)? Assume that 1 meter ≡ 3 feet.