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Abstract—Unmanned aircraft systems (UAS) generally use Global Navigation Satellite System (GNSS) measurements to estimate their state (position and orientation) for outdoor navigation. However, in urban environments, GNSS pseudorange measurements contain biases due to multipath effects and signal blockages by nearby buildings. For safe navigation in such environments, it is beneficial to predict the state uncertainty while accounting for the effect of measurement biases. Reachability analysis is a commonly used tool to predict the state uncertainty of a system. However, existing works do not account for the effect of measurement biases on state estimation, which consequently affects the predicted state uncertainty. Additionally, majority of the existing literature focuses on linear systems, whereas the dynamics of widely used practical systems are better captured by non-linear models. Thus, in this paper we present a non-linear stochastic reachability analysis to predict bounds on the state uncertainty while accounting for measurement biases. We derive the analysis for a fixed-wing UAS navigating using ranging measurements. In order to evaluate our predicted bounds for GNSS-based navigation, we simulate a 3D urban environment and account for the effect of measurement biases on state estimation. Methods to predict the state uncertainty distribution along a planned trajectory have been previously addressed in literature [14]–[19]. The underlying assumption in these works is that the measurement distributions along the planned trajectory is known in advance. However, such an assumption is not always valid for GNSS-based navigation. GNSS pseudorange measurement biases in urban areas are difficult to predict in advance, thus, resulting in uncertain measurement distributions along the trajectory. The effect of these uncertain measurement distributions need to be accounted for while predicting the state uncertainty.

Reachability analysis provides a powerful formal verification-based tool to predict bounds for future state uncertainty [20]. Classical reachability analysis consists of computing a set of future reachable states for a system, given an initial set of states and set of system inputs [21]. Existing works primarily address the problem of computing exact reachable sets for linear systems [22]–[25]. However, the dynamics of widely used practical systems such as fixed-wing UAS are better captured by non-linear models [26], thus, requiring a non-linear reachability analysis. While there exist works that compute conservative reachable sets for non-linear systems [27]–[29], these do not account for the effect of uncertain measurement distributions on the state estimation and consequently on the reachable set. Thus, in this paper, we present a non-linear stochastic reachability analysis to predict the state uncertainty bounds while accounting for uncertain GNSS pseudorange measurement distributions as illustrated in Fig. 1.

I. INTRODUCTION

RECENTLY there has been growing interest in outdoor unmanned aircraft systems (UAS) applications for various purposes such as delivering goods, surveying areas of interest, and search and rescue [1]–[5]. The Federal Aviation Administration (FAA) projects the UAS fleet within the United States to grow to 1.4 million vehicles by the year 2023 [6]. Thus, in the future the airspace will likely be shared by multiple UAS performing various operations. For autonomous outdoor navigation, Global Navigation Satellite System (GNSS) is commonly used to estimate the UAS state (position and orientation) [7], [8]. However, operations might involve navigating through urban areas, where GNSS pseudorange measurements contain additional biases due to signals being reflected or blocked by nearby structures. These are classified either as multipath effects, where both the direct and reflected signals from the same satellite are received; or as non-line-of-sight (NLOS) effects, where only the reflected satellite signal is received [9]. Generally, NLOS effects result in large biases, and various outlier rejection techniques and 3-dimensional (3D) map-based techniques have been proposed to detect and exclude the corresponding pseudorange measurements [10]–[12]. On the other hand, biases due to multipath effects are relatively lower, the bounds for which can be calculated given the 3D map and the GNSS receiver architecture [9], [13]. These measurement biases affect the UAS state estimate, which consequently affects the UAS state uncertainty.

To enable safe GNSS-based navigation in urban areas, it is beneficial to not only address the pseudorange measurement biases on-line, but also to predict the effect of these biases on future UAS states. Methods to predict the state uncertainty distribution along a planned trajectory have been previously addressed in literature [14]–[19]. The underlying assumption in these works is that the measurement distributions along the planned trajectory is known in advance. However, such an assumption is not always valid for GNSS-based navigation. GNSS pseudorange measurement biases in urban areas are difficult to predict in advance, thus, resulting in uncertain measurement distributions along the trajectory. The effect of these uncertain measurement distributions need to be accounted for while predicting the state uncertainty.

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The main contributions of the paper are listed as follows:

1) We extend our previous work [30] to present a non-linear stochastic reachability analysis to predict the state uncertainty bounds, while accounting for sets of uncertain measurement distributions.

2) In the non-linear stochastic reachability analysis, we provide a method to approximate the Lagrange remainders (linearization errors) with Gaussian distributions.

3) Next, we apply the presented analysis to a fixed-wing UAS navigating using GNSS pseudorange measurements and derive the corresponding equations required for the analysis.

4) Finally, we simulate an urban environment for GNSS-based navigation. We calculate the bias bounds for GNSS pseudorange measurements due to multipath effects. Multiple simulations are performed to validate that our predicted state uncertainty bounds enclose future UAS states with a desired confidence level. Additionally, we also show the applicability of our predicted bounds to enable safe navigation for fixed-wing UAS in a shared airspace.

The rest of the paper is organized as follows: we begin by discussing related work in section II; in section III we formulate our problem; section IV provides set representations used in the paper along with the required operations; in section V we present all details for our non-linear stochastic reachability analysis including the state estimation process and the approximation of the Lagrange remainders; and in section VI we demonstrate our simulation results for urban GNSS-based navigation for a fixed-wing UAS.

II. RELATED WORK

A. State Estimation with Uncertain Measurement Distributions

Generally state estimation algorithms assume one of two distinct measurement error models: a stochastic error model or a bounded error model. Typically for stochastic error models the measurement errors are assumed to have a Gaussian distribution and the state is estimated using Bayesian filters such as the Kalman filter [31]. On the other hand, set-membership techniques such as the zonotopic Kalman filter [32] have been proposed for bounded error models. In [33], [34] the authors merge the two models by considering the sum of a Gaussian distribution and a bounded bias for the measurement errors. Such a merged approach is suitable for modeling errors from uncertain measurement distributions, where the measurement biases are not known in advance. However, these works focus on designing optimal state estimators and do not predict the state uncertainty, which is of our primary interest.

B. Predicting State Uncertainty

Uncertainty-aware trajectory planning algorithms [14]–[19] typically involve predicting the state uncertainty along candidate trajectories in order to provide probabilistic collision-free guarantees. In [14], [15] the authors predict the state uncertainty for a robot with linear-quadratic-Gaussian (LQG) control, while accounting for sensing uncertainties. [16] predicts a distribution over the robot states by considering the state estimation error distribution as well as all possible state estimates that could be realized in the future. In [17], [18] the authors predict the state uncertainty for a robot with linear-quadratic-Gaussian (LQG) control, while accounting for sensing uncertainties. [16] predicts a distribution over the robot states by considering the state estimation error distribution as well as all possible state estimates that could be realized in the future. In [17], [18] the authors assume the measurement errors to have known Gaussian distributions. In [17] the authors assume known Gaussian distributions for measurements associated with visual features in the stored map. [18] also assumes a Gaussian distribution for the localization uncertainty which is obtained using photometric information available in advance. However, such an assumption cannot be applied for GNSS pseudorange measurements, where the measurement distributions depend on details of the surrounding 3D structures as well as receiver-specific characteristics.

C. Reachability Analysis

Reachability analysis is a verification tool that is commonly used to provide safety guarantees of dynamical systems. The basic idea is to predict bounds for future states that the system can reach (referred to as reachable sets) and ensure that these bounds do not contain any unsafe states. Initial reachability analysis works [22]–[25] computed the reachable sets for linear systems given an initial set and a set of possible control inputs. Extensions to non-linear systems [27]–[29], [35], [36] were explored in two primary directions. In [28], [29] the authors linearized a non-linear system, and computed the reachable sets for the linearized system while computing conservative bounds for the linearization errors referred to as Lagrange remainders. An alternate approach
was taken in [35], [36], where the authors used polynomial set representations and computed the reachable sets directly for a non-linear system. However, such set representations come with an additional computational cost [37] which is not desirable for purposes such as trajectory planning. Different flavors of reachability analysis have been explored trajectory planning. In [37]–[39] the authors use a reachability toolbox [40] to compute reachable sets for all possible trajectories, and select trajectories where the reachable sets do not intersect with obstacles. [41]–[43] follow a similar approach, where [41] uses a polynomial set representation and [42], [43] use level sets for their reachability analysis. These works typically assume that the system has access to its true state along the trajectory. However, such an assumption is not always valid since measurement errors result in state estimation errors, which consequently affect the reachable sets.

III. PROBLEM FORMULATION

The following discrete-time non-linear motion model is considered for our system:

\[ x_k = f(x_{k-1}, u_{k-1}) + w_k, \]  

(1)

where \( x_k \) is the state vector, \( u_k \) is the input vector, \( f \) is a function representing the non-linear dynamics, \( w_k \) is the process noise vector modeled as a Gaussian distribution \( \mathcal{N}(0, Q) \), and \( k \) represents the time instant. We consider non-linear measurements of the form:

\[ z_k = h(x_k) + b_k + \epsilon_k, \]  

(2)

where \( z_k \) is the measurement vector, \( h \) is a function representing the non-linear measurement model, \( b_k \) is the bounded measurement bias vector and \( \epsilon_k \) is the measurement noise vector modeled as a Gaussian distribution \( \mathcal{N}(0, R_k) \). We assume that a trajectory from an initial state \( \hat{x}_0 \) to a final state \( \hat{x}_T \) has been planned, and that the following information is available along the trajectory:

1) Initial state estimation error covariance \( P_0 \).
2) Sets of nominal states \( X^{0:T} = \{\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_T\} \) and nominal inputs \( U^{0:T-1} = \{\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{T-1}\} \), such that:

\[ \hat{x}_k = f(\hat{x}_{k-1}, \hat{u}_{k-1}) \quad \forall \; k \in [1, T]. \]  

(3)

3) Set of stabilizing linear state feedback control gains \( K^{0:T} = [\hat{K}_0, \hat{K}_1, \ldots, \hat{K}_T] \), such that the total control input is of the form:

\[ u_k = \hat{u}_k - \hat{K}_k(\hat{x}_k - \hat{x}_k). \]  

(4)

where \( \hat{x}_k \) is the on-line state estimate.
4) Sets of measurement bias vector bounds \( B^{0:T-1} = (B_0, B_1, \ldots, B_{T-1}) \) and measurement noise covariance matrices \( R^{0:T-1} = (R_0, R_1, \ldots, R_{T-1}) \).

The planning of such a trajectory along with computing the corresponding sets \( \hat{X}, \hat{U}, \hat{K} \) has been widely addressed in literature [31], [44] and is beyond the scope of this paper. For example, the optimal nominal trajectory for a Dubins model can be easily obtained in closed form [45]; and the feedback control gains can be obtained using a locally optimal linear-quadratic regulator (LQR) design.

Let \( \phi(x_k) \) denote the state uncertainty distribution at time instant \( k \) along the trajectory. From equations (1) and (4) we observe that the state \( x_k \) depends on the state estimate \( \hat{x}_{k-1} \) via the applied feedback control. The state estimate in turn depends on the set of received measurements. Thus, each different set of measurement biases \( \hat{\phi}^{0:k-1} \in B_0 \times \cdots \times B_{k-1} \) results in a different set of measurement distributions (equation (2)), consequently resulting in a different state uncertainty distribution \( \phi(x_k) \).

We define our problem as: given a trajectory along with the required information, predict bounds \( \mathcal{R} \) such that they enclose all possible state uncertainty distributions:

\[ \max_{\phi(x_k)} \Pr(x_k \notin \mathcal{R}_k) < \delta, \; \forall \; k \in [0, T], \]  

(5)

where \( \delta \) is a threshold obtained from a desired confidence level. Thus, for a \( m \sigma \) confidence level, where \( m > 0 \), \( \delta \) is calculated as: \( \delta = 1 - e^{-\tau(m/\sqrt{2})} \). Fig. 2 illustrates the defined problem.

IV. SET REPRESENTATIONS AND OPERATIONS

For our stochastic reachability analysis we use probabilistic zonotopes as the set representation. Zonotopes and probabilistic zonotopes have been shown to be computationally efficient and closed under linear transform and Minkowski sum operations [21]. A zonotope \( \mathcal{P} \) is defined as follows:

\[ \mathcal{P} = \left\{ x \in \mathbb{R}^n | x = c_p + \sum_{i=1}^{r} \beta_i \cdot g_p^{(i)}, -1 \leq \beta_i \leq 1 \right\}, \]  

(6)

where \( c_p \) is the center of the zonotope, and \( g_p^{(i)} \) are \( n \)-dimensional column vectors referred to as generators. The generators of a zonotope determine its shape relative to its center. The zonotope can be concisely written as \( \mathcal{P} = \mathcal{Z}(c_p, G_p) \), where \( G_p = [g_p^{(1)}, \ldots, g_p^{(r)}] \) is the corresponding \( n \times r \) generator matrix. Fig. 3(a) shows an example 2D zonotope along with its generator matrix. Extending zonotopes to a stochastic framework, a probabilistic zonotope is defined as a Gaussian distribution with an uncertain mean and can be written as:

\[ \mathcal{P} = \mathcal{Z}(c_\mathcal{P}, G_\mathcal{P}, \Sigma_\mathcal{P}), \]  

(7)
where \( c_\mathcal{P} \) and \( G_\mathcal{P} \) represent the zonotope bounding the uncertain mean, and \( \Sigma_\mathcal{P} \) is the Gaussian covariance of the probabilistic zonotope. Fig. 3(b) provides an example visualization of a 2D probabilistic zonotope. Note that probabilistic zonotopes do not have a normalized distribution, and in fact represent probabilistic hulls that can enclose multiple distributions.

The Minkowski sum operation between two probabilistic zonotopes is defined as [21]:
\[
\mathcal{P}_1 \oplus \mathcal{P}_2 = \mathcal{Z}(c_\mathcal{P}_1 + c_\mathcal{P}_2, [G_\mathcal{P}_1, G_\mathcal{P}_2], \Sigma_\mathcal{P}_1 + \Sigma_\mathcal{P}_2),
\]
whereas, the linear transform operation is defined as [21]:
\[
T \cdot \mathcal{P} = \mathcal{Z}(Tc_\mathcal{P}, TG_\mathcal{P}, T\Sigma_\mathcal{P}^T).
\]

Additionally, we define the following function to calculate the projection of a probabilistic zonotope along a column vector \( e \) for a \( m\sigma \) confidence level:
\[
\text{proj}(\mathcal{P}, e, m) = \sum_{i=1}^{p} |e^\top \cdot g^{(i)}_\mathcal{P}| + m \sqrt{e^\top \cdot \Sigma_\mathcal{P} \cdot e},
\]
where the first term represents the projection of the uncertain mean, and the second term represents the projection of the covariance matrix.

V. STOCHASTIC REACHABILITY ANALYSIS

In this section, we present the different components of our non-linear stochastic reachability analysis. We first describe the state estimation filter used on-board. Next, we provide the details of our stochastic reachability analysis and how we predict the state uncertainty bounds \( \mathcal{R} \). Finally, we explain our method to approximate the Lagrange remainder terms that arise due to linearizations within the reachability analysis.

A. Extended Kalman filter (EKF)

During autonomous flight, the true state \( x_k \) will not be available to compute the control input. Thus, we need to obtain an estimate of the state \( \hat{x}_k \) and compute the total input as shown in equation (4). Given the non-linear motion and measurement models, we use an EKF as the state estimation filter. The prediction step of the EKF is performed as:
\[
\begin{align*}
\hat{x}_k &= f(\hat{x}_{k-1}, u_{k-1}), \\
\hat{P}_k &= A_kP_{k-1}A_k^\top + Q,
\end{align*}
\]
where \( A_k = \frac{\partial f}{\partial x} \bigg|_{x_k=\hat{x}_k} \) and \( Q \) is the covariance of the process noise defined in equation (1). The measurements available to the filter can be re-written from equation (2) as follows:
\[
z_k = h(x_k) + c_{b_k} + \nu_k,
\]
where \( c_{b_k} = (\bar{b}_k - \tilde{b}_k)/2 \) is the deterministic center of the set \( B_k \), and \( \nu_k \) is modeled as a Gaussian distribution with an uncertain mean, i.e.:
\[
\nu_k \sim \mathcal{N}(b_k - c_{b_k}, R_k), \quad b_k \in B_k.
\]

For the EKF correction step, we choose an over-bounding hypothesis \( \hat{R}_k \) as the measurement covariance matrix. The over-bounding is performed such that \( \hat{R}_k \) matches the tail of the distribution in (14) at a \( m\sigma \) confidence level. Thus, the covariance for each measurement \( \hat{R}_k^{(i)} \) is computed as:
\[
\hat{R}_k^{(i)} = \left[ \left( \frac{w_{b_k} + m\sqrt{\hat{R}_k^{(i)}}}{m} \right)^2 \right]_i,
\]
where \( w_{b_k} = (\bar{b}_k - c_{b_k}) \) is the half-width vector of the set \( B_k \), and the superscript \( (i) \) refers to the \( i^{th} \) element for a vector and to the \( i^{th} \) diagonal element for a matrix. Fig. 4 illustrates the hypothesis for a single measurement. Note that our stochastic reachability analysis does not necessarily require choosing an over-bounding hypothesis for the EKF measurement covariance matrix. If desired, a different hypothesis can be chosen and used with the rest of the analysis. We choose the over-bounding hypothesis since it is equivalent to scaling or inflating the covariance matrix, which is a commonly used approach for practical implementation of KF and its variants [46]–[48].
Once $\hat{R}_k$ has been computed, the EKF correction step is performed as:

$$L_k = P_k C_k^T (C_k P_k C_k^T + R_k)^{-1},$$

$$\hat{x}_k = \bar{x}_k + L_k (z_k - h(\bar{x}_k) - c_o)$$

$$P_k = P_k - L_k C_k P_k,$$

where $L$ is the Kalman gain and $C_k = \frac{\partial h}{\partial x}\big|_{x=\bar{x}_k}.$

### B. Prediction of state uncertainty bounds

Using our stochastic reachability analysis, we compute stochastic reachable sets $\mathcal{X}$ along the trajectory. These sets are enclosing hulls, which provide a distribution over all the states that might be reached in the presence of measurement biases. The required state uncertainty bounds $\mathcal{R}$ are computed as the projections of $\mathcal{X}$ associated with a $\sigma$ confidence level.

We first formulate the equations governing the growth of a single state vector, and later shift to a set notation. To keep the notations concise, we define a combined state and input vector $s^T = [x^T, u^T]$. Linearizing equation (1) about the planned trajectory:

$$x_k = f(\hat{s}_k-1) + w_k + \frac{\partial f(s)}{\partial x}\big|_{s=\hat{s}_k-1} (s_k-1 - \hat{s}_k-1) + \frac{1}{2} (s_k-1 - \hat{s}_k-1)^T \frac{\partial^2 f(s)}{\partial x^2}\big|_{s=\hat{s}_k-1} (s_k-1 - \hat{s}_k-1) + \ldots,$$

and considering the first-order approximation of the Taylor series, we get:

$$x_k = f(\hat{s}_k-1) + w_k + \frac{\partial f(s)}{\partial x}\big|_{s=\hat{s}_k-1} (s_k-1 - \hat{s}_k-1) + \frac{1}{2} (s_k-1 - \hat{s}_k-1)^T \frac{\partial^2 f(s)}{\partial x^2}\big|_{s=\hat{s}_k-1} (s_k-1 - \hat{s}_k-1),$$

where $\xi \in \{\hat{s}_k-1 + \alpha(s_k-1 - \hat{s}_k-1) | \alpha \in [0, 1]\}$ if $s_k-1$ is restricted to a convex set and if $\hat{s}_k-1$ and $\hat{s}_k-1$ are fixed [28], [49]. Here the last term of equation (20) is the vector of Lagrange remainders resulting from linearizing the function $f$ w.r.t. the vector $\hat{s}_k-1$ about $s_k-1$. Splitting $s$, the above equation can be written as:

$$x_k = A_{k-1}(x_k-1 - \hat{x}_k-1) + B_{k-1}(u_k-1 - \hat{u}_k-1) + \frac{\partial f(s)}{\partial x}\big|_{s=\hat{s}_k-1} (s_k-1 - \hat{s}_k-1) + \frac{1}{2} (s_k-1 - \hat{s}_k-1)^T \frac{\partial^2 f(s)}{\partial x^2}\big|_{s=\hat{s}_k-1} (s_k-1 - \hat{s}_k-1),$$

where $B_k = \frac{\partial f}{\partial u}\big|_{u=\hat{u}_k}$, and $\mathcal{L}_{[s,s]k-1}$ is a concise notation for the Lagrange remainder vector. Using equations (3) and (4) to substitute the nominal state and the total input, we get:

$$x_k = A_{k-1}(x_k-1 - \hat{x}_k-1) - B_{k-1}K_{k-1}(\hat{x}_k-1 - \hat{x}_k-1) + \hat{x}_k + \mathcal{L}_{[s,s]k-1} + w_k.$$  

We define the on-line state estimation error as $\tilde{x}_k = \hat{x}_k - x_k$, and re-write the above equation as:

$$x_k = (A_{k-1} - B_{k-1}K_{k-1})(x_k-1 - \hat{x}_k-1) - B_{k-1}K_{k-1}\tilde{x}_k-1 + \hat{x}_k + \mathcal{L}_{[s,s]k-1} + w_k.$$  

Expanding on the state estimation error, we use equations (13) and (17) to re-write $\tilde{x}_k$ as:

$$\tilde{x}_k = \bar{x}_k + L_k(h(x_k) + \nu_k - h(\bar{x}_k)) - x_k$$

The non-linear measurement model can be linearized w.r.t. $x_k$ and $\tilde{x}_k$ about the nominal state $\bar{x}_k$, to obtain the following:

$$h(x_k) = h(\bar{x}_k) + C_k(x_k - \bar{x}_k) + \mathcal{L}_{(x,x)k}^h,$$

$$h(\tilde{x}_k) = h(\bar{x}_k) + C_k(\tilde{x}_k - \bar{x}_k) + \mathcal{L}_{(\hat{x},x)k}^h,$$

where $\mathcal{L}_{(x,x)k}^h$ and $\mathcal{L}_{(\hat{x},x)k}^h$ represent the corresponding Lagrange remainder vectors. Thus, equation (24) can be written as:

$$\tilde{x}_k = (I - L_k C_k)(\bar{x}_k - x_k) + L_k(\mathcal{L}_{(x,x)k}^h - \mathcal{L}_{(\hat{x},x)k}^h) + L_k\nu_k.$$  

Thus, subtracting (21) from (28), the error in the predicted state can be written as:

$$\tilde{x}_k - x_k = A_{k-1}(\hat{x}_k-1 - x_k) + B_{k-1}(u_k-1 - \hat{u}_k-1) + f(\hat{x}_k-1, \hat{u}_k-1) + \mathcal{L}_{[s,s]k-1} - \mathcal{L}_{[s,s]k-1} - w_k.$$  

On substituting equation (29) in equation (27) we obtain the following equation for the state estimation error:

$$\bar{x}_k = (I - L_k C_k)A_{k-1}\hat{x}_k-1 + (I - L_k C_k)(\mathcal{L}_{[s,s]k-1} - \mathcal{L}_{[s,s]k-1}) + L_k(\mathcal{L}_{(x,x)k}^h - \mathcal{L}_{(\hat{x},x)k}^h) - (I - L_k C_k)w_k + L_k\nu_k.$$  

The growth of the state vector is governed by equations (23) and (30). However, since the noise quantities are random and not deterministic, there is a set of possible states $\mathcal{X}$ that can be reached along the trajectory. These sets depend on the sets of possible noise quantities. From their definitions in equations (1) and (13), we recall that $w_k \sim \mathcal{N}(0, Q)$ and $\nu_k \sim \mathcal{N}(b_k - c_{o_k}, R_k)$. Using the set notations described in section IV, we define:

$$\mathcal{W}_k = \mathcal{Z}(0, 0, Q),$$

$$\mathcal{V}_k = \mathcal{Z}(0, \text{diag}(w_{b_k}), R_k).$$

Thus, using the sets of noise quantities, equation (30) can be adapted to obtain the set of possible on-line state estimation errors:

$$\bar{x}_k = (I - L_k C_k)A_{k-1}\hat{x}_k-1 \oplus (I - L_k C_k)(\mathcal{L}_{[s,s]k-1} \oplus \mathcal{L}_{[s,s]k-1}) + L_k(\mathcal{L}_{(x,x)k}^h \oplus \mathcal{L}_{(\hat{x},x)k}^h) \oplus (I - L_k C_k)\mathcal{W}_k \oplus L_k\mathcal{V}_k.$$  

and similarly, equation (23) can be adapted to obtain the stochastic reachable sets:

$$\bar{x}_k = (A_{k-1} - B_{k-1}K_{k-1})(\hat{x}_k-1 - \hat{x}_k-1) \oplus B_{k-1}K_{k-1}\hat{x}_k-1 + \hat{x}_k \oplus \mathcal{L}_{[s,s]k-1} \oplus \mathcal{W}_k.$$  

(34)
Here we assume the sets $X_k$ and $\bar{X}_k$ to be independent while performing the Minkowski sum operation. The initial sets are formed using the initial state and the initial state estimation uncertainty as: $X_0 = (0, 0, P_0)$ and $\bar{X}_0 = (\bar{x}_0, 0, P_0)$. Thus, we use equations (33) and (34) iteratively to compute the stochastic reachable sets $X_k$. Finally, the state uncertainty bound along any direction $e$ is computed as:

$$\mathcal{R}_k = \text{proj}(X_k, e, m).$$ \hspace{1cm} (35)

C. Lagrange remainder approximation

In this section, we explain the steps that we take for approximating a general Lagrange remainder vector $L_{[p, \bar{p}]}$ resulting from linearizing the function $q$ w.r.t. $p$ about the point $\bar{p}$. The $i$th element of the Lagrange remainder vector can be defined as:

$$L_{[p, \bar{p}]}^{(i)} = \frac{1}{2} (p - \bar{p})^\top J_p^{(i)} (\xi) (p - \bar{p}),$$ \hspace{1cm} (36)

where $J_p^{(i)} (\xi) = \frac{\partial q_p^{(i)}(\xi)}{\partial p}$. Here $p$ belongs to a stochastic set $\mathcal{P}$, thus resulting in a stochastic distribution for the Lagrange remainder $L_{[p, \bar{p}]}^{(i)}$. In order to determine $L_{[p, \bar{p}]}^{(i)}$ in an efficient way, the following Gaussian approximation is computed:

1) First the $m\sigma$ projection $\mathcal{P}^{m\sigma}$ set for $p$ is obtained. The projection of $\mathcal{P}$ along each dimension is calculated as follows:

$$\bar{p}^{(i)} = \text{proj}(\mathcal{P}, e_i, m),$$ \hspace{1cm} (37)

where $e_i$ represents a unit column vector with the $i$th element set to 1. $\mathcal{P}^{m\sigma}$ is then set as:

$$\mathcal{P}^{m\sigma} = \mathcal{Z}(e\mathcal{P}, \bar{p}).$$ \hspace{1cm} (38)

2) Using the approach presented in [28], the corresponding maximum value for the Lagrange remainder is calculated:

$$L_{[p, \bar{p}]}^{(m\sigma)} = \frac{1}{2} \gamma^\top \max(|J_p^{(i)} (\xi(p))|) \gamma,$$

with $p \in \mathcal{P}^{m\sigma}$ and $\gamma = |c\mathcal{P} - \bar{p}| + \sum_{i=1}^{n} |\bar{p}^{(i)}|$. \hspace{1cm} (39)

3) The Lagrange remainder is approximated with a $m\sigma$ confidence level as a Gaussian distribution $\mathcal{N}(0, \Sigma_{L_p^{(i)}})$, where the covariance is determined as:

$$\Sigma_{L_p^{(i)}} = \frac{\left( L_{[p, \bar{p}]}^{(m\sigma)} \right)^2}{m}.$$ \hspace{1cm} (40)

This method is followed to approximate each of the Lagrange remainders in equations (33) and (34).

VI. URBAN GNSS-BASED FIXED-WING UAS NAVIGATION

In this section, we describe our simulation setup used to validate the predicted state uncertainty bounds. We first provide details of the fixed-wing UAS motion and measurement models, and the corresponding matrices required for the Lagrange remainder approximation. Next, we show our simulated 3D urban environment and our method to obtain the multipath bias bounds for the GNSS pseudorange measurements. Finally, we discuss the results obtained by predicting the state uncertainty bounds along multiple planned trajectories.

A. Fixed-wing UAS motion and measurement models

We consider a 2D scenario and model the fixed-wing UAS dynamics with a Dubins model [31]. The state vector consists of the 2D position $(x_1, x_2)$ and the heading angle $\theta$. The inputs to the system are the forward velocity $V$ and the angular velocity $\omega$. Thus, the motion model of the system is represented as:

$$\begin{bmatrix}
    x_{1k} \\
    x_{2k} \\
    \theta_{k}
\end{bmatrix} = \begin{bmatrix}
    x_{1k-1} \\
    x_{2k-1} \\
    \theta_{k-1}
\end{bmatrix} + \begin{bmatrix}
    V_{k-1} \cos(\theta_{k-1}) \Delta t \\
    V_{k-1} \sin(\theta_{k-1}) \Delta t \\
    \omega_{k-1} \Delta t
\end{bmatrix} + w_k,$$

where $\Delta t$ is the time-step we set as 0.2 s. We set covariance matrix for $w_k$ as $Q = \text{diag}([0.01 m^2, 0.01 m^2, 0.001 rad^2])$. For the state estimation filter, we use GNSS pseudorange measurements along with heading measurements from an on-board compass. The GNSS pseudorange measurement from the $i$th satellite can be expressed as [9]:

$$\rho_k^{(i)} = r(x_k, x_k^{(i)}) + b_k^{(i)} + c\delta t + \epsilon_k^{(i)},$$ \hspace{1cm} (42)

where $r$ represents the true range between the receiver position $x_k$ and satellite position $x_k^{(i)}$ (which we simulate from publicly available almanac data), $b_k^{(i)}$ is the additional bias caused by multipath/NLOS effects, $c\delta t$ is the clock bias error and $\epsilon_k^{(i)} \sim \mathcal{N}(0, \Sigma^{(i)})$. Since we are primarily concerned with the UAS position states, we assume for simplicity that the receiver clock and the satellite clocks are perfectly synced, i.e. there is zero clock bias error. However, if desired, clock bias states can also be included in the state vector for the stochastic reachability analysis. We model the noise covariance with an elevation-based factor [50], [51] as: $\Sigma^{(i)} = \Sigma_{\rho}/\sin^2(e^{(i)})$, where we set $\Sigma_{\rho} = 5 m^2$. Thus, given $N$ GNSS satellites, the measurement model for the fixed-wing UAS looks as follows:

$$\begin{bmatrix}
    z_k^{(1)} \\
    \vdots \\
    z_k^{(N)} \\
    z_k^{(N+1)}
\end{bmatrix} = \begin{bmatrix}
    r(x_k, x_k^{(1)}) \\
    \vdots \\
    r(x_k, x_k^{(N)}) \\
    h(x_k)
\end{bmatrix} + \begin{bmatrix}
    b_k^{(1)} \\
    \vdots \\
    b_k^{(N)} \\
    0
\end{bmatrix} + \begin{bmatrix}
    \epsilon_k^{(1)} \\
    \vdots \\
    \epsilon_k^{(N)} \\
    \epsilon_k^{(N+1)}
\end{bmatrix},$$ \hspace{1cm} (43)

where $z_k^{(i)} = \rho_k^{(i)} \forall i \in [N]$, and $z_k^{(N+1)}$ represents the heading measurement. To approximate the Lagrange remainders as shown in section V-C, the following double derivative matrices for the non-linear dynamics are obtained:

$$J^{(2)} = \begin{bmatrix}
    0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 \\
    0 & -V \cos(\theta) \Delta t & -\sin(\theta) \Delta t & 0 & 0 \\
    0 & -\sin(\theta) \Delta t & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{bmatrix},$$ \hspace{1cm} (44)
steps:

point along the planned trajectory, we perform the following
the pseudorange multipath bias bounds for each satellite at any
and assume that the reflected signal strength can be as strong
possibility of strong signal reflections from nearby buildings,
between the early and late correlators [9]. We consider the
× 1 km in Unity game engine. The environment dimensions are
B. Simulation setup and multipath bounds
For our simulations, we setup a large 3D urban environment
Fig. 5: Multipath noise envelope for a receiver with a quarter-
chip early/late correlator spacing [9].

where s is the combined state and input vector. Similarly, the
double derivative matrices for the non-linear measurements
are:

\[
J_{s}^{f_2} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -V \sin(\theta) \Delta t & \cos(\theta) \Delta t & 0 & 0 \\
0 & \cos(\theta) \Delta t & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad (45)
\]

\[J_{s}^{f_3} = 0_{5 \times 5}, \quad (46)\]

where s is the combined state and input vector. Similarly, the
double derivative matrices for the non-linear measurements
are:

\[
J_{x}^{f_2} = \begin{bmatrix}
\frac{(x_2-x_1)^2}{(r(x,x^2))^2} & -\frac{(x_2-x_1)(x_2-x_1^2)}{(r(x,x^2))^3} & 0 \\
\frac{(x_2-x_1)-1}{(r(x,x^2))^2} & \frac{(x_2-x_1)^2}{(r(x,x^2))^3} & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad (47)
\]

\[
J_{x}^{h_2(N+1)} = 0_{3 \times 3}. \quad \quad (48)
\]

B. Simulation setup and multipath bounds
For our simulations, we setup a large 3D urban environment
in Unity game engine. The environment dimensions are 1 km
× 1 km wide, and contain buildings up to 120 m tall. For the
GNSS receiver architecture, we assume a quarter-chip spacing
between the early and late correlators [9]. We consider the
possibility of strong signal reflections from nearby buildings,
and assume that the reflected signal strength can be as strong
as the direct LOS signal. Based on these characteristics, we
get a multipath noise envelope as shown in Fig. 5. To obtain
the pseudorange multipath bias bounds for each satellite at any
point along the planned trajectory, we perform the following
steps:

1) Identify possible reflecting surfaces as shown in Fig. 6.
We assume that the receiver is able to detect and exclude
any signals arriving from below the receiver [52], [53].
Thus, buildings lower than the flight altitude are ignored.

2) Calculate the possible differential path lengths for each
surface (including diffuse reflections) and store the overall
minimum differential path length.

3) Use the multipath noise envelope to obtain the pseudor-
range bias bounds by considering all possible differential
path lengths greater than or equal to the overall mini-

Fig. 6: Simulated 3D urban environment, along with ray-
tracing performed to compute the differential path lengths for
multipath signals.

C. Validating predicted bounds
We consider multiple planned trajectories in our simulated
urban environment to validate the predicted state uncertainty
bounds. From equation (35), we compute \( R_{k}^{e_k} \) and \( R_{k}^{h_k} \)
associated with a 3\( \sigma \) confidence level (i.e. \( m = 3 \)). Here \( e_{k}^{\perp} \)
and \( e_{k}^{\parallel} \) represent the unit vectors perpendicular and parallel to
the planned trajectories respectively:

\[ e_{k}^{\perp} = [-\sin(\hat{\theta}_k) \cos(\hat{\theta}_k) 0]^T, \quad (49) \]

\[ e_{k}^{\parallel} = [\cos(\hat{\theta}_k) \sin(\hat{\theta}_k) 0]^T. \quad (50) \]

Next, for each planned trajectory we simulate 1000 trajectories
and compare the perpendicular and parallel state errors with
the corresponding predicted bounds. To validate the bounds,
we check the number of simulated trajectories enclosed within
the bounds. For a 3\( \sigma \) confidence level, the predicted bounds
should enclose approximately 997 out of the 1000 trajectories

Fig. 7-10 show the results on a simplified representation for
clearer visualization. Buildings shaded light orange are
below the flight altitude and do not contribute to any multipath
biases in the pseudorange measurements, whereas buildings
shaded dark orange contribute to the multipath bias bounds as
discussed previously in section VI-B. In Fig. 7 we simulate
open-sky conditions by setting the UAS flight altitude to
125 m. We observe that the predicted 3\( \sigma \) bounds enclose 996
out of the 1000 simulated trajectories. This demonstrates that
our non-linear reachability analysis is suitable to predict state
uncertainty bounds in the absence of any measurement biases,
similar to previous works [14], [16], [17].

Next, in Fig. 8 we consider multiple planned trajectories
at lower flight altitudes of 65 m, where GNSS pseudorange
measurements might contain biases due to multipath. We
observe again that the predicted bounds enclose the simul-
ated trajectories reflecting the desired 3\( \sigma \) confidence level.
Compared to the open-sky trajectory, our predicted bounds
tend to over-approximate the state uncertainty in presence of
measurement biases. This happens because of two reasons.
First, the Lagrange remainder approximation tends to over-
approximate the tail probabilities in the presence of biases.
And second, at each time instant along the trajectory, our
stochastic reachability analysis considers all of the possible
biases in the pseudorange measurements. Hence, the com-
puted stochastic reachable sets along the planned trajectory


Fig. 7: Predicted state uncertainty bounds for a fixed-wing UAS in an open-sky environment (absence of multipath effects). (a) $3\sigma$ bounds (black) and 1000 simulated trajectories (blue) along with corresponding (b) perpendicular and (c) parallel state errors. The predicted bounds enclose 996 out of the 1000 trajectories, thus, reflecting the desired $3\sigma$ confidence level.

Fig. 8: Predicted state uncertainty bounds for a fixed-wing UAS in the presence of uncertain GNSS multipath biases. The predicted $3\sigma$ bounds (black) enclose 996, 994 and 995 trajectories respectively out of the 1000 simulated trajectories (blue) in each scenario, thus, reflecting the desired $3\sigma$ confidence level.
We have presented a non-linear stochastic reachability analysis to predict UAS state uncertainty bounds along a planned trajectory. We accounted for non-linear measurements with Gaussian distributions and bounded biases. The formulation for navigating a fixed-wing UAS using GNSS pseudorange measurements was derived. We setup a 3D urban simulation environment to simulate GNSS pseudorange measurement biases due to multipath effects, and validated that our predicted bounds enclosed future UAS state reflecting a desired confidence level. In the presence of measurement biases, we observed that the predicted bounds tend to over-approximate the state uncertainty. In practice, the over-approximation would lead to a more conservative evaluation of safety of a planned trajectory. Finally, we demonstrated how the predicted bounds can be used to ensure UAS safety while navigating in a shared airspace. A future direction of work is to explore alternate efficient approximations for the Lagrange remainders. Another promising direction includes using stochastic reachability analysis to predict state uncertainty bounds within a trajectory planning framework.

VII. CONCLUSION

We have presented a non-linear stochastic reachability analysis to predict UAS state uncertainty bounds along a planned trajectory. We accounted for non-linear measurements with...


