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Determining the Speed of Light by Measuring the Beat Frequency between Adjacent Axial Modes of a He-Ne Fabry-Pérot Laser

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Physics
Determining the speed of light by measuring the beat frequency between adjacent axial modes of a He–Ne Fabry-Pérot laser

We determine the speed of light in air to be \( c = (2.99 \pm 0.02) \times 10^8 \text{ m/s} \) by measuring the beat frequency between adjacent axial modes of a He–Ne Fabry-Pérot laser. A fast photodetector connected to a spectrum analyzer was used to measure the beat frequency. We also study the frequency pulling effect and conclude that it does not affect the calculated value of \( c \).

I. INTRODUCTION

Fabry-Pérot cavity was invented in 1897 [1] and since then it has been frequently used in many high-precision measurements, for example, in the attempt to detect gravitational waves [2, 3], and more recently in measuring fractional charge [4]. In this experiment, we study axial modes of a He–Ne Fabry-Pérot laser and beat frequencies between adjacent modes. In particular, we investigate the effects of frequency pulling and thermal drift [10] on the beat frequency. We then calculate the speed of light in air from the measured beat frequency.

Due to doppler broadening [9], the emission profile of a He–Ne laser follows a Gaussian lineshape centered at \( \nu_0 = 474 \text{ THz} \) (633 nm) with a full width at half maximum (FWHM) \( \Delta \nu_0 \) (see Fig. 1a). The typical length of a He–Ne Fabry-Pérot cavity is about 50 cm, so the laser only oscillates at frequencies that are integer multiples of \( \approx 316 \text{ MHz} \). Hence, several axial modes are present in the Fabry-Pérot cavity, at positions indicated by the vertical blue lines in Fig. 1a.

However, lasing only occurs when the intensity of the axial mode is above the gain threshold [5] (see the horizontal dotted black line in Fig. 1a). In this experiment, only two axial modes \( n \) and \( n + 1 \) are above the threshold and they are amplified through stimulated emissions [6]. As other axial modes are very low in intensity, we effectively only observe two modes in the laser output (see Fig. 1b).

When these two adjacent axial modes interfere, the resulting intensity at the photodetector, which is the square of sum of two sinusoidal electric fields, will have an oscillating component at frequency equal to \( \nu_{\text{FSR}} = \nu_{n+1} - \nu_n \), where \( \nu_{\text{FSR}} \) is the free spectral range. This is the beating phenomenon. As derived in Appendix A, \( \nu_{\text{FSR}} \) is linearly related to the variation of cavity length \( \Delta L \) by:

\[
\frac{1}{\nu_{\text{FSR}}} = -\frac{2}{c} \Delta L + b \tag{1}
\]

where \( c \) is the speed of light in air, and \( b \) is a constant incorporating \( c \) and the optical path length of the Fabry-Pérot cavity. From Eq. 1, we can calculate \( c \) by measuring \( \Delta L \) and determining \( \nu_{\text{FSR}} \) from the beat frequency.

However, the actual beat frequency \( \nu_b \) is smaller than \( \nu_{\text{FSR}} \) due to the frequency pulling effect [10]. This is because the index of refraction of He–Ne gas in the discharge tube increases with photon frequency. This means light at higher frequency has a slightly longer optical path length, hence a lower
resonant frequency. As a result, each axial mode $\nu_n$ (vertical blue lines in Fig. 1a) is pulled towards the central frequency $\nu_a$ by an amount $\delta \nu_n$ given by [10]:

$$
\delta \nu_n \simeq -\frac{g(\nu_n - \nu_a)}{2\pi \Delta \nu_a} \nu_{FSR} \quad (2)
$$

where $g$ is the power gain per round trip, $\Delta \nu_a$ is the FWHM of the Gaussian lasing profile, and the negative sign in front indicates that frequencies are pulled towards $\nu_a$. As derived in Appendix B, $\nu_{FSR}$ is related to the measured beat frequency $\nu_b$ by:

$$
\nu_{FSR} \simeq \nu_b + \frac{g \nu_b^2}{2\pi \Delta \nu_a}. \quad (3)
$$

Hence, we need to add a correction term to the measured value of $\nu_b$ to obtain $\nu_{FSR}$. In this experiment, the correction term is about 0.05 MHz, which is small compared to $\nu_b \simeq 316$ MHz. As analyzed in Sec. III, the correction to $\nu_{FSR}$ in fact does not affect the calculated value of $c$.

II. METHODOLOGY

To determine $c$, we need to measure how beat frequency $\nu_b$ between the two lasing axial modes (see Fig. 1b) varies with the change in cavity length $\Delta L$. We placed a He–Ne discharge tube inside a Fabry-Pérot cavity, and we varied the cavity length by changing the position of its output coupler via a micrometer, whose precision is 5 µm. The laser output was measured by a fast photodetector, which was connected to a radio frequency (RF) spectrum analyzer. The spectrum analyzer is able to measure frequencies in the range of 9 kHz–3 GHz, which includes our expected beat frequency $\nu_b \simeq 316$ MHz.

As beat frequencies are different between different transverse modes of the laser [10], we placed an iris inside the Fabry-Pérot cavity and reduced its size until only the TEM$_{00}$ mode was present. In addition, Brewster windows at two ends of the He–Ne discharge tube ensured all axial modes were of the same polarization.

As expected, we observed a single peak near 316 MHz in the spectrum analyzer, which corresponds to the beat frequency between the $n$-th and the $(n+1)$-th axial modes of the laser (see Fig. 1b). To reduce fluctuations in the power spectral density of the peak, we used the video averaging option in the spectrum analyzer. Fig. 1c shows a typical plot of power spectral density $S$ against frequency $\nu$ for beat frequency $\nu_b = 316.31$ MHz. Note the distribution in Fig. 1c is just the difference between the two axial mode intensity distributions in Fig. 1b. We determined the peak position in Fig. 1c by using the peak search function in the spectrum analyzer.

Due to thermal drift in the Fabry-Pérot cavity [10], the power spectral density profile in Fig. 1c drifted slightly in the horizontal direction in the time scale of five seconds, giving rise to an observed uncertainty of 0.03 MHz in the peak position. This uncertainty is our dominant source of statistical error in the determination of $c$ (see Sec. III).

III. RESULTS AND ANALYSIS

We measured 29 beat frequencies $\nu_b$ by varying the output coupler position $\Delta L$ at intervals of 0.127 mm. As discussed in Sec. I, due to the frequency pulling effect, $\nu_{FSR}$ is slightly larger than $\nu_b$ by $\delta \nu_b = \frac{g \nu_b^2}{4\pi^2 \nu_{FSR}}$ (see Eq. 3). We first estimate the value of $\delta \nu_b$. From Ref.[8], $\Delta \nu_a = (18140 \pm 900)$ MHz. From Ref.[7], the single-pass gain per unit length of the He–Ne discharge tube for 633nm line is $(13 \pm 4)$% m$^{-1}$. The round trip path length in the discharge tube is $(0.378 \pm 0.001)$ m [12], so $g = 0.049 \pm 0.015$. For a typical $\nu_b = 316$ MHz in this experiment, $\delta \nu_b = (0.051 \pm 0.015)$ MHz, which is on the same order of magnitude with the uncertainty in the measured value of $\nu_b$, which is 0.03 MHz.

Since $\delta \nu_b$ is small, the effect of this correction on the plot of $\nu_{FSR}^{-1}$ against $\Delta L$ is a downward translation of all data points by the same amount equal to $\frac{g \nu_b^2}{4\pi^2 \nu_{FSR}}$ (see Appendix C for derivation). Hence, the correction due to frequency pulling will not affect the gradient, or $-2/c$ (see Eq. 1). We have performed linear regressions with and without the correction $\delta \nu_b$, and we have verified that both give us identical values for $c$ within its uncertainty.

Fig. 2 shows a plot of $\nu_{FSR}^{-1}$ against $\Delta L$ without the correction due to frequency pulling. As mentioned in Sec. II, the error bars arise from the thermal drift of the Fabry-Pérot cavity, and they correspond to an uncertainty of $\pm 0.03$ MHz in $\nu_{FSR}$. The red line shows a linear fit and the lower subplot shows the residuals.

The residuals display a clear systematic trend: they are below zero near the center of the plot but above zero at the two ends. We suspect the systematic trend is a result of non-linearities in the optics used in this experiment, including two Brewster windows, two cavity mirrors and a high-reflection mirror that directs the laser beam to the photodetector. As the principle of superposition does not hold in non-linear optics, the beat frequency is not just the difference between two adjacent axial modes, but...
FIG. 2: Plot of the reciprocal of the free spectral range $\nu_{FSR}^{-1}$ against change in the Fabry-Pérot cavity length $\Delta L$. The errorbars in $\nu_{FSR}$ arise from the thermal drift of the Fabry-Pérot cavity [10]. The red line shows a linear fit and the gradient is $-2/c$ (see Eq. 1). The lower subplot shows the residuals, which display a clear systematic trend.

contains second or higher order terms involving the frequency of each axial mode. Hence, we observe second and higher order non-linear trend in the data points, which give rise to a systematic pattern in the residuals. Nonetheless, further investigation is necessary to reproduce and explain this trend.

The reduced $\chi^2$ value is 1.16 for our linear fit. From Ref.[13], this reduced $\chi^2$ value indicates that about 25% of time our repeated experiments will yield the above result with our estimated uncertainties, assuming Eq. 1 is correct. However, given the systematic trend in the residuals, the small $\chi^2$ value suggests that the 0.03 MHz uncertainty in $\nu_b$ is overestimated. This is because although the peak position for $\nu_b$ (see Fig. 1c) drifted horizontally back and forth with an uncertainty of 0.03 MHz, we consistently determined the peak position for $\nu_b$ when the profile in Fig. 1c drifted to the equilibrium position in the center. Hence, the uncertainty in $\nu_b$ is less than 0.03 MHz.

By Eq. 1, the slope in Fig. 2 is equal to $-2/c$. We hence estimate the speed of light in air to be:

$$c = (2.99 \pm 0.02) \times 10^8 \text{ m/s}$$

where the uncertainty is derived from the uncertainty in the gradient parameter of the linear fit.

IV. CONCLUSION

We have determined the speed of light in air to be $c = (2.99 \pm 0.02) \times 10^8$ m/s by measuring the beat frequency $\nu_b$ between adjacent axial modes of a He–Ne Fabry-Pérot laser. We also consider the frequency pulling effect on $\nu_b$ but conclude that the effect is negligible and it does not affect the calculated value of $c$.

The most dominant error in this experiment is the statistical error associated with the measurement of beat frequency $\nu_b$. It is due to the thermal drift of the Fabry-Pérot cavity. Therefore, to further increase the precision of the measurement for $c$, we can lower the temperature of the Fabry-Pérot cavity and decrease the power of laser output. Besides, to account for the non-linear behavior of $\nu_{FSR}^{-1}$ as a function of $\Delta L$, we shall consider the effect of non-linear optics in the derivation of Eq. 1, and repeat the experiment to verify the revised relation between $\nu_{FSR}^{-1}$ and $\Delta L$.

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APPENDIX A

We derive Eq. 1 in this section. From Ref. [12],

$$\frac{1}{\nu_{FSR}} = \frac{2[L + (n_{avg} - 1)d]}{c}$$

where $L$ is the physical length of the Fabry-Pérot cavity, $d$ is the laser path length in materials other than air, $n_{avg}$ is the average index of refraction of these materials, and $c$ is the speed of light in air.

We can write $L = L_o - \Delta L$, namely, a fixed component $L_o$ and a variable component $\Delta L$. Hence,

$$\frac{1}{\nu_{FSR}} = -\frac{2\Delta L}{c} + \frac{2[L_0 + (n_{avg} - 1)d]}{c}$$

and the second term on the right hand side consists only of constants $L_o$, $d$, $c$ and $n_{avg}$. Thus, we denote this constant term by $b$ and Eq. 1 follows.

APPENDIX B

We derive Eq. 3 in this section. Referring to Fig. 1a, the original frequency of axial modes (the
vertical blue lines) is

\[ \nu_n = n \nu_{\text{FSR}}. \]  

(4)

Due to frequency pulling, \( \nu_n \) is displaced by \( \delta \nu_n \) as given by Eq. 2. Hence, the new positions of the pulled frequencies are

\[ \nu'_n = \nu_n + \delta \nu_n = \nu_n - \frac{g(n \nu_{\text{FSR}} - \nu_a)}{2\pi \Delta \nu_a} \nu_{\text{FSR}} \]

\[ = n \nu_{\text{FSR}} - \frac{g(n \nu_{\text{FSR}} - \nu_a)}{2\pi \Delta \nu_a} \nu_{\text{FSR}} \]  

(5)

where we have substituted \( \nu_n \) using Eq. 4. The actual beat frequency is

\[ \nu_b = \nu'_n - \nu'_{n-1} = \nu_{\text{FSR}} \left(1 - \frac{g \nu_{\text{FSR}}}{2\pi \Delta \nu_a}\right). \]

Last, since the correction term \( \frac{g \nu_{\text{FSR}}}{2\pi \Delta \nu_a} \) is small, we approximate \( \nu_{\text{FSR}} \approx \nu_b \), which yields Eq. 3.

**APPENDIX C**

We will explain how an addition of \( \delta \nu_b \) on \( \nu_b \) will affect the value of \( y = 1/\nu_b \), assuming \( \delta \nu_b \) is small. Differentiating \( y \) with respect to \( \nu_b \) yields

\[ dy = -\frac{1}{\nu_b^2} d\nu_b. \]

Since \( \delta \nu_b \) is small, we can thus approximate \( \delta \nu_b \approx d\nu_b \). But from Eq. 3,

\[ \delta \nu_b = \frac{g \nu_b^2}{2\pi \Delta \nu_a} \]

we thus have

\[ \delta y = -\frac{1}{\nu_b^2} \delta \nu_b = -\frac{g}{2\pi \Delta \nu_a}. \]

This indicates that if \( \delta \nu_b \) is small, adding \( \delta \nu_b \) to \( \nu_b \) will effectively decrease the value of \( y = 1/\nu_b \) by a fixed amount of \( \frac{g}{2\pi \Delta \nu_a} \).

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[5] The author would like to thank N. Breznay for pointing this out.
[6] To be precise, other axial modes may also exceed the gain threshold but we only empirically observed two dominant lasing axial modes. This fact can be explained by the spatial hole burning effect (see Ref. [10]) in which only two adjacent axial modes are dominant. This is because the gain competition between the two adjacent axial modes is much reduced when each mode uses different groups of atoms for oscillation.
[8] The measurement of \( \Delta \nu_a \) was performed using a spectrometer connected to a photomultiplier in Lab 2 of Phy. 107 at Stanford University (2013).