Symplectic geometry of scattering diagrams for log CY surfaces

James Pascaleff

University of Illinois at Urbana-Champaign
Supported by NSF DMS-1522670

Mirror Symmetry and Wall-Crossing, Berkeley, March 21, 2016
1. Log CY surfaces
2. Floer-theoretic reconstruction
3. Asymptotic disk counts
4. Consequences
Definition
A (exact) log CY surface (with maximal boundary) is a pair \((Y, D)\)

- \(Y\) smooth projective surface,
- \(D\) nodal anticanonical divisor, which supports an ample divisor.

Thus \(U = Y \setminus D\) is an exact symplectic manifold.
Example: \(Y =\) cubic surface, \(D =\) triangle of lines.
The divisor $D$ consists of a cycle of $\mathbb{P}^1$’s. We can arrange that the symplectic structure is \textit{locally toric} near $D$. Fibers of local moment maps give Lagrangian tori $L_b$ near $D$. $L_b$ has Maslov class zero in $U$. 
Reconstruction

[Fukaya, Tu, Abouzaid-Auroux-Katzarkov] For generic $J$, $L_b$ bounds no disks (since space of disks with boundary on $L_b$ has virtual dimension $\mu + n - 3 = -1$), and its Floer theory is undeformed. The space of objects supported on $L_b$ is therefore identified with $H^1(L_b, U_\Lambda) \cong U_{\Lambda}^2$.

$$\Lambda = \left\{ \sum_{i=1}^{\infty} c_i T^{r_i} \mid c_i \in \mathbb{C}, r_i \in \mathbb{R}, \lim_{i \to \infty} r_i = \infty \right\}$$

$$U_\Lambda = \left\{ c_0 + \sum_{i=1}^{\infty} c_i T^{r_i} \mid c_0 \neq 0, r_i > 0 \right\} \subset \Lambda$$
Reconstruction

As $L_b$ moves, glue together $H^1(L_b, U^\wedge)$ by a combination of rescalings and maps induced by pseudoisotopies of $A_\infty$-structures.

- Rescalings: $(L_t, J_t)$, $L_t$ non-hamiltonian deformation, but $J_t$ chosen so that no disks appear. This just rescales the coefficients.
- Pseudoisotopies: $(L_t, J_t)$, $L_t$ hamiltonian deformation, but as $J_t$ varies Maslov 0 disks may appear; in this parametrized problem they have dimension $\mu + n - 3 + 1 = 0$.

The pseudoisotopies realize a wall-crossing phenomenon (walls of the first kind).
Reconstruction

- Let \((z_1, z_2) \in U^2_\Lambda\) be coordinates from an identification \(H^1(L_0, U_\lambda) \cong U^2_\Lambda\).

- In the case where all disks in the family \((L_t, J_t)\) lie in multiples of a particular class \(\beta \in H_2(X, L_0)\), the pseudoisotopy acts as a transformation \(z_i \mapsto h_i(z_\beta)z_i\), where \(h_i(z) \in 1 + z\mathbb{Q}[[z]]\) and \(z_\beta = T^{\omega(\beta)}z^{[\partial \beta]}\).
Basic problem

So, we “just” need to count the Maslov 0 disks. But these counts themselves are not stable: if we deform the path \((L_t, J_t)\), the Maslov 0 disk counts may change (walls of the second kind). A \(J_t\)-disk can break into a \(J_{t'}\)-disk and a \(J_{t''}\)-disk. (But sphere bubbling cannot occur due to exactness of \(U\).) [Cf. Yu-Shen Lin]
Tori near boundary

Take path $b(t)$ in base of torus fibration near $D$, and consider a path of Lagrangian tori $L_{b(t)}$ close to the boundary divisor $D$. We want to apply Fukaya’s procedure to this family of tori. After a small reformulation, we need to count $\mu = 0$ disks with boundary on $L_{b(t)}$ for some $t$. 
Lemma

The moduli spaces of $\mu = 0$ disks stabilize as the path $b(t)$ approaches the boundary of the moment map image.

- That is to say, for each relative homotopy class $\beta \in H_2(X, L_{b(t)})$, if we take the path $b(t)$ close enough to the boundary, eventually no wall-crossing of the second kind occurs.

- Morally, this is related to the idea that the GHK scattering diagram is an asymptotic object.
Stabilization

- As the path of tori $L_{b(t)}$ collapses onto the boundary divisor, holomorphic disks must approach holomorphic spheres in $Y$ (target-local Gromov compactness [J.W. Fish]).
- By successively blowing up nodes of $D$, one can assume that the disks in the class $\beta$ we are interested in are limiting to spheres that intersect the smooth part of $D$. Eventually, no bifurcations can occur because disks in classes $\beta_1, \beta_2$ such that $\beta = \beta_1 + \beta_2$ are separated from each other.
Relative invariants

- Let \((\tilde{Y}, \tilde{D})\) be a toric blow up of \((Y, D)\), \(C\) a component of \(\tilde{D}, \tilde{D}' = \tilde{D} \setminus C\). Set \(\tilde{Y}^0 = Y \setminus \tilde{D}', C^0 = C \cap \tilde{Y}^0\).
- Let \(\beta \in H_2(\tilde{Y}, \mathbb{Z})\) be a class such that \(\beta \cdot C = k\beta\) and \(\beta \cdot (\text{each other component}) = 0\).
- Consider space of relative maps \(\overline{M}(\tilde{Y}/C, \beta)\) with full tangency of order \(k\beta\) and an unspecified point of \(C^0\), and the subspace \(\overline{M}(\tilde{Y}^0/C^0, \beta)\) of maps whose image lies in \(\tilde{Y}^0\).

Lemma (Gross-Pandharipande-Siebert, GHK)
\(\overline{M}(\tilde{Y}^0/C^0, \beta)\) is compact.

- Define \(N_\beta = \#\overline{M}(\tilde{Y}^0/C^0, \beta)\) to be the virtual count of curves.
- The GHK scattering diagram is defined on an singular affine manifold \(B\) in terms of \(N_\beta\). The rays correspond to components in some toric blow up.
Scattering diagram

\[ f(z) = \exp \left[ \sum_{\beta} k_\beta N_\beta T^{\omega(\beta)} z^{[\partial\beta]} \right], \quad z_i \mapsto f(z)^{n_i} z_i \]
Symplectic sum

- To understand disks in $\tilde{Y}^0$, perform a symplectic cut on a neighborhood of $C^0$, thus writing $\tilde{Y}^0 = \tilde{Y}^0 \# C^0 \mathbb{P}(N_{C^0} \oplus \mathbb{C})$.
- Perform the cut in such a way that a piece of the Lagrangian boundary condition $L_t$ remains in the $\mathbb{P}(N_{C^0} \oplus \mathbb{C})$ factor.
- A symplectic sum formula relates the parameterized moduli spaces of disks with boundary on $L_t$ to (relative) moduli spaces of disks in $\mathbb{P}(N_{C^0} \oplus \mathbb{C})$ and moduli spaces of closed curves in $\tilde{Y}^0$.
- The closed curves are precisely the ones used in the definition of the GHK scattering diagram!

[Li-Ruan, Ionel-Parker, Jun Li]
[Nishinou-Siebert, M. Farajzedeh Tehrani]
Symplectic sum
Compatibility

- Since the disk counts are not entirely stable, it is important to understand whether different degenerations are compatible.
- To deal with several divisor components (↔ rays) simultaneously, perform toric blow-ups to make the divisors we are interested in disjoint.
- We can treat this situation as symplectic sum along a disconnected divisor.
Compatibility

- In this way, we can show that the disk counts match the GHK formula for any particular finite collection of rays in the scattering diagram.
- Work order by order in Novikov parameter $T$: To a particular order, all but finitely many rays are trivial.
- Thus, by working with more and more “refined” degenerations, we can calculate the wall-crossing to any desired order, and match it to the GHK description.
- Thus, for families of Lagrangian tori $L_b(t)$ sufficiently close to the boundary divisor, the wall crossing transformations are given by the functions in the GHK scattering diagram.
Symplectic cohomology

- A Floer cohomology for Liouville manifolds, such as $U = Y \setminus D$.
- Complete $U$ to $\hat{U}$ by attaching a positive half of a symplectization $\Sigma \times [0, \infty)$, $\Sigma \subset U$ a contact type hypersurface.
- Let $H$ be a Hamiltonian that grows like $cr^2$ on the end, define $SH^*(U) = HF(H)$. (Actually involves perturbation of $H$.)
- $SH^*(U)$ detects classical topology of $U$ and the closed Reeb orbits in $\Sigma$, in a deformation-invariant way.
Symplectic cohomology
Basis of SH

- In the log CY surface case, $\Sigma$ is a $T^2$-bundle over $S^1$ with nontrivial monodromy.
- The Reeb flow is tangent to the $T^2$ fibers, and translates each one by a certain vector $\nu(t) \in \mathbb{R}^2$. One can arrange that the vector $\nu(t)$ rotates monotonically as we vary the fiber.
- When $\nu(t)$ has rational slope, periodic orbits appear. These orbits and their iterates contribute non-trivial cocycles to symplectic cohomology.

**Theorem (P. 2013)**

*There is a basis for the degree zero symplectic cohomology $SH^0(U)$ indexed by the integral points of the GHK affine manifold $B$.***
Basis of SH

\[ B(\mathbb{Z}) \]
Closed-open map

- The representation of elements of $SH^0(U)$ as functions on the mirror is furnished by the closed-open map counting half-cylinders asymptotic to $SH$ generators with boundary on Lagrangian tori, weighted by area and holonomy.

$$SH^0(U) \to \mathcal{O}(H^1(L, U_\lambda)).$$

- For any $p \in B(\mathbb{Z})$, we have a divisor component $C_p$ (in some toric blow up $(\tilde{Y}, \tilde{D})$), we have a torus $L_p$ and a generator $\theta_p \in SH^0(U)$ linking that divisor.

- In some situations, we can argue that the closed open map sends $\theta_p$ to a monomial in $\mathcal{O}(H^1(L_p, U_\lambda))$. (One which is invariant w.r.t. wall-crossing along $C$).
Closed-open map

- From here on, we can follow GHK formally.
- Image of $\theta_p$ in other charts $\leftrightarrow$ broken lines.
- To determine the coefficient of $\theta_r$ in $\theta_p \cdot \theta_q$, evaluate in the chart corresponding to $L_r$, and take the coefficient of the corresponding monomial.
Conclusion

• Construction of the mirror is governed by $\mu = 0$ disks, the counts of which stabilize in the limit as the tori collapse onto the divisor.

• These asymptotic disk counts are related to relative GW invariants for closed $g = 0$ curves via a symplectic sum formula.

• This gives us enough information to construct the mirror “at infinity,” and to understand other structures, such as $SH^0(U)$, that also live “at infinity.”
Thank you!