The Flux Reconstruction Approach to High-Order Methods: Theory and Application

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Introduction

The flux reconstruction (FR) approach is efficient, simple to implement, and allows various high-order schemes, such as the nodal discontinuous Galerkin (DG) method and any spectral difference (SD) method, to be cast within a single unifying framework [1]. Recently, we have identified a new class of 1D linearly stable FR schemes [2]. Identification of such schemes offers significant insight into why certain FR schemes are stable, whereas others are not. Also, from a practical standpoint, the resulting formulation offers a simple prescription for implementing an infinite range of intuitive and linearly stable high-order methods. We are currently extending the 1D formulation to multiple dimensions (including to simplex elements). We are also developing CPU/GPU enabled linearly stable high-order methods. We are currently extending these schemes to multiple dimensions (including to simplex elements).

Flux Reconstruction

- Consider solving the following 1D scalar conservation law within a standard element $r = [-1, 1]$, where $f$ is a flux of $u$

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial r} = 0$$

- The basic steps of a FR approach (that utilizes second-order solution polynomials) are outlined in Figure 1.

Energy Stable Flux Reconstruction

- The form of the correction functions is critical
- For linear stability of a FR scheme (that utilizes order $k$ solution polynomials) one requires

$$\int r^i \gamma_i \, dr = \frac{0}{ck!} \left( \frac{d^{k+1}g_k}{d\gamma^{k+1}} \right)_{i = k-1}^{i = k-2}$$

$$\frac{-2}{(2k + 1)!(a_k k!)^2} < c < \infty$$

$$a_k = \frac{(2k)!}{2^{2k}(k!)^2}$$

- These conditions are satisfied if

$$g_k = \frac{(-1)^k}{2} \left( \frac{\eta_k}{L_k} - \left( \frac{\eta_{k-1} + L_{k+1}}{1 + \eta_k} \right) \right)$$

$$\eta_k = \frac{c(2k + 1)(a_k k!)^2}{2}$$

$$L_k = \frac{1}{2k!} \left( \frac{d^k}{dr^k} \left[ r^2 - 1 \right]^k \right)$$

- They imply stability in a Sobolev type norm of the form

$$\left[ \left| \int_1^0 (\hat{u}^2) \, dx \right| \right]_{k,2} = \left[ \left| \int_1^0 \left( \frac{\partial \hat{u}}{\partial \hat{\gamma}} \right)^2 \, dx \right| \right]^{1/2}$$

- Various existing schemes can be recovered

$$c = 0$$

$$c = 2k / \left( (2k + 1)(k + 1)(a_k k!)^2 \right)$$

$$c = 2(k + 1) / \left( (2k + 1)(a_k k!)^2 \right)$$

- Currently undertaking Fourier analysis to understand how properties of the schemes vary with $c$ (see Figure 2).

Implementation

- Currently developing compressible inviscid and viscous flow solvers based on linearly stable FR schemes
- Designed to work with unstructured 2D (triangular and quadrilateral) and 3D (hexahedral and tetrahedral) meshes
- Parallelized for multiple CPUs and multiple GPUs
- Almost linear scaling on multiple GPUs, running at 1.2 Teraflops (double precision) on 16 Tesla C2050 GPUs

Summary

- The FR approach unifies various well-known high-order methods within a single framework
- Recently, we have identified a range of 1D linearly stable FR schemes [2]
- We have extended these schemes to multiple dimensions (including to simplex elements)
- We are developing unstructured high-order compressible inviscid and viscous flow solvers based on the range of linearly stable FR schemes
- Solvers are parallelized for multiple CPUs and multiple GPUs

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References
