Energy Stable High-Order Methods for Compressible Viscous Flows

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### High-Order Methods

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- **Application:** Computational Fluid Dynamics
- **Motivation:** Higher-fidelity results

High-order methods ...
- Produce less numerical dissipation than traditional 2nd-order methods
- Capable of propagating vortex structures for longer distances
- Are more efficient for simulating certain unsteady flows
Unsteady Flows of Interest

Introduction  ESFR Procedure  Linear problems  Nonlinear problems  Conclusion

[1]  [2]  [3]  [4]

[1] [2] [3] [4]
Problem: Flexibility and ease of implementation

High-order methods ...
- Are less robust towards spurious oscillations, Gibbs phenomenon
- More difficult to implement than traditional 2nd-order methods
- Frequently require the implementation of complex quadrature procedures

Possible solutions ...
- The creation of new ‘Quadrature-free’ high-order methods
- Stabilizing strategies that improve robustness of high-order methods
- Discovered in 2007 by Huynh [5]
- Framework that unifies several well known high-order methods
  - Spectral Difference (SD), [6, 7]
  - Discontinuous Galerkin (DG), [8-11]
- Proposes new methods
  - Schemes with improved robustness
  - One particular schemes that has time-step limit that is 2x larger than time-step limit for collocation-based nodal DG scheme
Energy Stable Flux Reconstruction

- Abbreviated as ESFR
- Vincent-Castonguay-Jameson-Huynh (VCJH) schemes [12, 13]
- Provably stable for linear advection problems
  - Linear advection is a model for Euler equations
  - Stability proof in 1D on segments and in 2D on triangles [14]
- Recovers stable formulations of SD and DG schemes
Recently, ESFR schemes have been proven stable for linear advection-diffusion problems on
- Triangles and tetrahedra
- All orders of accuracy
- Arbitrary unstructured grids that are convenient for applications


- Solve the nonlinear advection-diffusion equation

\[ \frac{\partial u}{\partial t} + \nabla \cdot f(u, \nabla u) = 0 \]

- To simplify matters, rewrite as a first-order system

\[ \frac{\partial u}{\partial t} + \nabla \cdot f(u, q) = 0 \]

\[ q - \nabla u = 0 \]

- Obtain a solution to the system on the domain \( \Omega \)
Rewrite the advection-diffusion equation in each element

$$\frac{\partial u_k^D}{\partial t} + \nabla \cdot f_k^D = 0$$

$$\mathbf{q}_k^D - \nabla u_k^D = 0$$

Superscript $D$ indicates that the quantity is discontinuous.

Enable communication between elements by introducing continuous $u_k, f_k$

$$\frac{\partial u_k^D}{\partial t} + \nabla \cdot f_k = 0$$

$$\mathbf{q}_k^D - \nabla u_k = 0$$
Transform element from physical space to reference space

\[ \Omega_k, (x, y) \quad \leftrightarrow \quad \Omega_S, (\hat{x}, \hat{y}) \]

\[ \Theta_k(\hat{x}) \]

\[ (\Omega_3, k, v_3, k) \quad (\Omega_2, k, v_2, k) \quad (\Omega_1, k, v_1, k) \quad (\Omega_S, k, v_1, k) \quad (\Omega_S, k, v_2, k) \quad (\Omega_S, k, v_3, k) \]

\[ (-1, 1) \quad (1, -1) \quad (-1, -1) \]

\[ (1, 1) \]

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The Flux Reconstruction approach involves solving the following system of equations on the reference domain $\Omega_S$

\[
\frac{\partial \hat{u}^D}{\partial t} + \hat{\nabla} \cdot \hat{f} = 0
\]

\[
\hat{q}^D - \hat{\nabla} \hat{u} = 0
\]

Equivalently,

\[
\frac{\partial \hat{u}^D}{\partial t} + \hat{\nabla} \cdot \hat{f}^D + \sum_{f=1}^{3} \sum_{l=1}^{3} \left( \hat{f}_{f,l}^C \cdot \hat{n}_{f,l} \right) \phi_{f,l} = 0
\]

\[
\hat{q}^D - \hat{\nabla} \hat{u}^D - \sum_{f=1}^{3} \sum_{l=1}^{3} \left( \hat{u}_{f,l}^C \cdot \hat{n}_{f,l} \right) \psi_{f,l} = 0
\]
An example of a correction function $g_{f,l}$ and associated correction field $\psi_{f,l}$ where

$$\psi_{f,l} = \hat{\nabla} \cdot g_{f,l}$$
Correlation Fields for ESFR schemes

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- Vincent-Castonguay-Jameson-Huynh (VCJH) correction fields (originally defined in [14]) for advection-diffusion problems in 2D and 3D
- Allow for $\phi_{f,l}$ to be parameterized by a single constant $c$, and for $\psi_{f,l}$ to be parameterized by a single constant $\kappa$
- The overall schemes are parameterized by only two constants
- Satisfy certain symmetry and orthogonality conditions
VCJH correction fields ensure stability of the schemes for linear advection-diffusion problems

\[
\frac{\partial u}{\partial t} + \nabla \cdot (au) - \nabla \cdot (b \nabla u) = 0
\]

First order form

\[
\frac{\partial u}{\partial t} + \nabla \cdot (au) - \nabla \cdot (b q) = 0
\]

\[
q - \nabla u = 0
\]

`Energy stability`

\[
\frac{1}{2} \sum_{k=1}^{N} \left( \frac{d}{dt} \|u_k^D\|_{p,c}^2 \right) + \sum_{k=1}^{N} (b\|q_k^D\|_{p,\kappa}^2) \leq 0
\]
The value of $c$ that yields the largest value of $\Delta t'_{\text{lim}}$ is denoted by $c_+$.
Nonlinear Numerical Experiments: SD7003 Geometry

- Flow over SD7003 airfoil and infinite wing
- Infinite wing simulated with finite span of $0.2c$, and periodic boundary conditions

- Subject to extensive numerical [16-20] and experimental [21,22] testing in 2D and 3D
Unsteady solution to the Navier-Stokes equations

Discretized with the following VCJH schemes
- \( c = c_{dg}, \kappa = \kappa_{dg} \)
- \( c = c_{+}, \kappa = \kappa_{+} \)

Simulated with
- Lax-Friedrichs formulation for advective numerical flux [23, 24]
- LDG formulation for diffusive numerical flux [25]
- Runge-Kutta, low-storage, 5 stage, 4th order time-stepping scheme [26]

Obtained 1.5 – 2.0x time-step limit increase for scheme with
\( c = c_{+}, \kappa = \kappa_{+} \)
Results obtained on unstructured triangular mesh with $N = 25810$ elements and an unstructured tetrahedral mesh with $N = 711332$ elements.
SD7003 Geometry

- Time averaged values of the lift and drag coefficient (2D)

<table>
<thead>
<tr>
<th>Source</th>
<th>$Re = 10K$</th>
<th>$Re = 60K$</th>
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<tbody>
<tr>
<td>$C_L$</td>
<td>$C_D$</td>
<td>$C_L$</td>
</tr>
<tr>
<td>Uranga et al. [20]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c_{dg}$, $\kappa_{dg}$</td>
<td>0.3755</td>
<td>0.5730</td>
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<tr>
<td>$c_+$, $\kappa_+$</td>
<td>0.3719</td>
<td>0.5831</td>
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<td>0.01975</td>
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<tr>
<td></td>
<td>0.04935</td>
<td>0.02005</td>
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- Time averaged values of the lift and drag coefficient (3D)

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<td>$c_+$, $\kappa_+$</td>
<td>0.3743</td>
<td>0.3454</td>
</tr>
<tr>
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<td>0.04967</td>
<td>0.04903</td>
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Density and vorticity contours of the flow with $Re = 10000$ and $M = 0.2$ around the SD7003 airfoil at $\alpha = 4^\circ$. Results were obtained on the unstructured triangular grid with $N = 25810$ elements using the VCJH scheme with $c = c_+, \kappa = \kappa_+, \text{ and } p = 2$. 
Density and vorticity contours of the flow with $Re = 60000$ and $M = 0.2$ around the SD7003 airfoil at $\alpha = 4^\circ$. Results were obtained on the unstructured triangular grid with $N = 25810$ elements using the VCJH scheme with $c = c_+, \kappa = \kappa_+$, and $p = 2$. 

$Re = 60000$, $M = 0.2$, $\alpha = 4^\circ$, $N = 25810$, $c = c_+$, $\kappa = \kappa_+$, $p = 2$
Density and vorticity isosurfaces colored by Mach number for the flow with $Re = 10000$ and $M = 0.2$ around the SD7003 wing-section at $\alpha = 4^\circ$. Results were obtained on the unstructured tetrahedral grid with $N = 711332$ elements using the VCJH scheme with $c = c_+, \kappa = \kappa_+$, and $p = 3$. 
Summary

- **General contribution**
  - Discovered a new class of energy-stable high-order methods for compressible, viscous problems on triangles and tetrahedra

- **Contributions for linear problems**
  - Proved stability for linear advection-diffusion problems on triangles and tetrahedra
  - Demonstrated favorable performance of schemes on linear problems

- **Contributions for nonlinear problems**
  - Demonstrated favorable performance of schemes on nonlinear problems
Future Work

- Theory
  - Prove the stability of the schemes ESFR/VCJH schemes on quadrilateral and hexahedral elements
  - Extension of ESFR/VCJH schemes to pyramids

- Applications
  - Apply the schemes on triangles and tetrahedra to high Reynolds number flows
  - Evaluate how schemes behave in concert with LES turbulence model
  - Turbomachinery, rotorcraft blades, and high-lift systems
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Questions?
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