Motivation

Engineers are often interested in integral estimation, for example finding the expected value of a function with uncertain inputs.

\[ \mathbb{E}[f] = \int f(x)p(x)dx \]

When \( \mathcal{X} \) is high dimensional or function evaluations are computationally expensive, this integral is impossible to compute with quadrature techniques. As a substitute, Monte Carlo methods are used to estimate this integral rather than calculate it explicitly. It is desired to maximize the accuracy of the estimate while minimizing the number of function evaluations necessary.

Purpose

Simple Monte Carlo converges to the correct answer at a rate of \( \sigma f \sqrt{N} \) where \( \sigma f \) is the standard deviation of \( f \) and \( N \) is the number of samples. Fortunately, this convergence does not depend on the dimensionality of the function, but unfortunately the convergence is slow; to gain an additional digit of accuracy one hundred times more samples are required. One way to reduce the estimation error is to decrease the standard deviation of the function of interest. This can be accomplished by making a fit (such as a polynomial) to the sample points and then integrating that fit. However, if the fit is poor, doing so can actually increase the error, and often times even if the fit is good, bias is added. Stacked Monte Carlo (StackMC) uses machine learning techniques to combine fits to the data samples with Monte Carlo techniques to yield an unbiased estimate with greater accuracy than using either method alone.

The basic idea is to add and subtract the expected value of a fit to the function.

\[ f = \int f(x)p(x)dx = \alpha \hat{f} + \int (f(x) - \alpha \hat{f})p(x)dx \]

The function \( f - \alpha \hat{f} \) has lower variance than \( f \), and so a Monte Carlo estimate of this integral will be more accurate than an estimate of the original integral. This equation is still exact and so it is unbiased, and the free parameter \( \alpha \) can be tuned to match the accuracy of the fit to achieve optimal variance reduction.

StackMC Method

1) Generate \( N \) samples from a Monte Carlo method.
2) Choose a fitting algorithm \( g \) such that the expected value \( \mathbb{E}[g] \) can be evaluated analytically.
3) Divide the data into \( k \) testing and training groups (like in \( k \)-fold validation).
4) Train \( k \) fits, \( g_i \), on the training data in the \( i \)-th group.
5) Evaluate \( g_i \) for each of the fits.
6) For each group, predict the value at all of the testing points.
7) Set \( \alpha = \frac{\sigma_f}{\sigma_g} \), where \( \sigma_f \) is the variance of the function values and \( \sigma_g \) is the variance of the predictions at the held out data points.
8) Correct the expected value of all of the fits \( \mathbb{E}[g_i] \) for each fit using their correlation to the samples.
9) Take the average of all the corrected expected values. 

\[ \mathbb{E}[f] \approx \sum \frac{1}{k} \mathbb{E}[g_i] \]

Application – Future Aircraft Fuel Burn

Economon and Copeland use the Program for Aircraft Synthesis Studies (PASS), a conceptual aircraft design tool, to predict the fuel burn of future aircraft given certain assumptions about technology advancement in the 2020 and 2030 time frames. In their predictions for single-aisle aircraft in 2020, the authors model eight probabilistic variables representing different effects of improvements in aircraft technology (propulsion, structures, aerodynamics). Each of the variables is represented by a unique beta distribution. The authors generated 15,000 samples (each representing one optimized aircraft) from which they measured the expected fuel-burn metric and the standard deviation of the expected fuel-burn metric. An accuracy comparison of StackMC can be seen above.

Application – Sonic Boom Uncertainty Quantification

Colonno and Alonso recently created a new sonic boom propagation tool, SuBoom, and used it to analyze the robustness of several aircraft pressure signatures optimized for minimal boom noise. Unlike the aircraft design test case, the response surface for the sonic boom noise signature is not smooth; the output can vary significantly with slight adjustments to the input parameters. In their high-fidelity near-field case, they have 62 uncertain variables: 4 representing aircraft parameters such as cruise Mach number and roll angle, and 58 representing uncertainties in the near-field pressure signal. A third-order polynomial was used as the fitting algorithm, and an accuracy comparison of StackMC can be seen below.

Conclusions

In both of the example cases, StackMC performs at least as well as the better of MC and the fitting function, and for a range of sample points outperforms both. The computation time of the StackMC algorithm is negligible; there are significant improvements in accuracy for less computation time than the cost of one additional function evaluation. StackMC is a very generic method for post-processing of data samples to improve accuracy. It can be used by anyone trying to estimate an integral or expected value of a function where \( f(x) \) is known.

Bibliography

