Boardman-Vogt Resolutions of Generalized Props

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Abstract. The Boardman-Vogt resolution of a colored topological operad is an important construction in homotopy theory that dates back to the early 1970s. It has broad applications in homotopy theory, homotopical algebra, higher category theory, and even biology.

This monograph is a systematic study of the Boardman-Vogt resolutions of generalized props in symmetric monoidal categories. Generalized props include small enriched categories, colored (di)operads, colored half-props, colored prop(erad)s, colored wheeled operads, and colored wheeled prop(erad)s, among others. Taking full advantage of the graphical machinery developed in the authors’ previous work, the Boardman-Vogt resolution of a generalized prop is defined in one step as a coend indexed by a graphically constructed category. Even for operads this coend definition of the Boardman-Vogt resolution is new.

For a connected pasting scheme under the right homotopical setting, it is proved here that the Boardman-Vogt resolution is a cofibrant resolution and has other nice homotopical properties. This applies to, for example, (wheeled) properads, (wheeled) operads, and dioperads. For a shrinkable pasting scheme, it is proved that the bar resolution of a generalized prop is an example of the Boardman-Vogt resolution. In particular, this applies to wheeled (pr)operads and (di)operads. All of these are also true for colored cyclic operads and colored modular operads. Along the way, coherence theorems for colored cyclic operads and colored modular operads are obtained.

A relative version of the Boardman-Vogt resolution that applies to a map of generalized props is defined and is shown to provide a functorial factorization of the given map. Under suitable conditions, this factorization involves a cofibration followed by a weak equivalence. A relative version of the bar resolution is also defined and is shown to be an example of the relative Boardman-Vogt resolution when the pasting scheme is shrinkable.

Assuming minimal background, this monograph is aimed at both graduate students and experienced researchers. Plenty of pictures, examples, motivational discussion, and applications are included. To clarify different parts of the theory, categorical arguments are carefully separated from homotopical ones.
• The first author dedicates this book to Eun Soo and Jacqueline.
• The second author would like to dedicate this book to his lovely wife Terri and to his nearly perfect children Lizzie and Ben.
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Introduction

In our previous work [YJ15], the authors introduced and studied in details the concept of a generalized prop associated to a pasting scheme. With the appropriate pasting schemes, generalized props encompass small enriched categories, colored (di)operads, colored half-props, colored prop(erad)s, colored wheeled operads, and colored wheeled prop(erad)s, among others. Our definition of a pasting scheme allowed us to study these algebraic objects with greater generality and clarity at the same time. With minor modifications this framework also applies to colored cyclic operads and colored modular operads. See Chapter 12 and Chapter 13.

This monograph is an in-depth study of resolutions of generalized props in symmetric monoidal categories. We concentrate on a particularly simple and geometrically inspired resolution called the Boardman-Vogt resolution, the Boardman-Vogt construction, or the W-construction. In specific instances the Boardman-Vogt construction has been an integral part of homotopy theory since the early 1970s. For topological colored operads, this construction was originally introduced by Boardman and Vogt [BV72] to study homotopy invariance of algebraic structures, such as $E_n$-spaces for $1 \leq n \leq \infty$. See Chapter 2 for a brief review of this classical construction. For operads in a symmetric monoidal category, the Boardman-Vogt construction was studied in [BM06, BM07]. When restricted to operads, our Boardman-Vogt construction reduces to the ones in [BV72, BM06, BM07] up to isomorphisms.

Applications

The Boardman-Vogt resolution is important for several reasons. First, algebras over cofibrant generalized props behave nicely in a homotopical sense. For example, as originally proved in [BV72], if $O$ is a nice enough topological operad, then algebras over its Boardman-Vogt construction $WO$ are homotopy invariant in the sense that they can be transferred back and forth via suitable weak equivalences. This kind of transfer of an algebra structure is far from true for a general operad. Along the same lines, for a nice enough differential graded operad, its Boardman-Vogt construction plays a crucial role in deformations of operadic algebras and in the proof of Deligne’s Conjecture in [KS00]. The same nice homotopical property also holds for algebras over cofibrant generalized props, such as the Boardman-Vogt construction of a nice enough generalized prop. We will see examples of this sort in Chapter 8.

The Boardman-Vogt construction is also important in its own right. For example, it is shown in [BM03b] that the Boardman-Vogt construction of a small category provides a conceptual definition of higher homotopy operations that include higher Massey products, higher Whitehead products, and long Toda brackets.
The Boardman-Vogt construction of a small category is discussed in Examples 6.1.6 and 6.3.5. Their algebras are discussed in Section 8.3.

Moreover, the recent work in $BO\infty$ demonstrates that the Boardman-Vogt construction can be a useful organizational tool in biology. It is shown in $BO\infty$ that phylogenetic trees assemble to form a topological operad. This phylogenetic operad is closely related to the Boardman-Vogt construction of the topological commutative operad. For a general pasting scheme and a symmetric monoidal category, we will discuss the Boardman-Vogt construction of the corresponding commutative generalized prop in Examples 3.1.11, 3.4.11, and 3.5.14. We believe that when the ambient category is that of topological spaces and the pasting scheme is that of connected wheel-free graphs (corresponding to properads), this Boardman-Vogt construction is closely related to phylogenetic networks in biology.

Furthermore, the Boardman-Vogt construction provides a robust definition of the homotopy coherent nerve, which is a crucial part of a satisfactory theory of $\infty$-generalized props. For small categories the homotopy coherent nerve goes back to at least $CP86$, $CP97$, and plays a prominent role in the theory of $\infty$-categories in $Lur09$. The Boardman-Vogt constructions of the categories $[n] = \{0 < 1 < \cdots < n\}$, which are discussed in Examples 6.1.6, 6.3.5, 10.1.6, and 10.2.3, define the homotopy coherent nerve from simplicially-enriched categories to simplicial sets as well as its left adjoint. For operads using the Boardman-Vogt construction in $BM06$, the homotopy coherent nerve was used in $MW09$ as part of a theory of $\infty$-operads. This homotopy coherent nerve goes from simplicially-enriched operads to dendroidal sets, and its left adjoint is also defined via the Boardman-Vogt construction of operads generated by unital trees.

For more complicated generalized props, such as (wheeled) properads, the graphical category generalizing the ordinal number category $\Delta$ and the dendroidal category in $MW09$ was constructed in $HRY15$. We expect that in the context of $HRY15$ the Boardman-Vogt construction here will yield a satisfactory homotopy coherent nerve of a (wheeled) properad. The first step in this direction is achieved in Section 8.4. We will show there that, for a connected pasting scheme, our Boardman-Vogt construction of a graphically generated generalized prop is a cofibrant resolution. Further development of the homotopy coherent nerve of generalized props using our Boardman-Vogt construction is the subject of ongoing work of Philip Hackney, Marcy Robertson, and the first author.

**Boardman-Vogt Construction**

We now provide a brief description of our Boardman-Vogt construction of a generalized prop. Roughly speaking the Boardman-Vogt construction is an appropriately fattened version of the free generalized prop construction. For a pasting scheme $G$ and an ambient symmetric monoidal category $M$, a $G$-prop $P$ in $M$ has structure maps

$$\gamma^P_G : P[G] = \bigotimes_{v \in Vt(G)} P(v) \longrightarrow P_{\text{root}(G)} \otimes P_{\text{in}(G)}$$

for graphs $G$ in the pasting scheme, satisfying a unity axiom and an associativity axiom. For example, for operads $G$ runs through all unital trees, and for wheeled props $G$ runs through all wheeled graphs. In other words, if $UP$ is the underlying
family of objects in $\mathcal{P}$, then its free $\mathcal{G}$-prop has entries of the form

$$FUP(\mathcal{G}) = \bigsqcup_G \mathcal{P}[G],$$

in which $G$ runs through the graphs in the pasting scheme with the given profile $\mathcal{(}\mathcal{G}) = (\text{vtx}(G))$.

The free $\mathcal{G}$-prop construction is discrete in the sense that it is a coproduct of the $\mathcal{P}$-decorated graphs $\mathcal{P}[G]$. The idea of the Boardman-Vogt construction is to introduce a homotopy into the free $\mathcal{G}$-prop construction by decorating each ordinary internal edge in $G$ by a commutative segment $J$. For example, for topological spaces we can use the unit interval $[0, 1]$, and for simplicial sets we can use the simplicial interval $\Delta^1$. The purpose of the commutative segment $J$ is to allow collapsing of sub-graphs in $G$ corresponding to graph substitution in the pasting scheme.

Using the structure of a commutative segment $J$ and the $\mathcal{G}$-prop structure of $\mathcal{P}$, the operation of collapsing sub-graphs provides many relations $\sim$ among the decorated graphs. Taking all of these relations into account, the Boardman-Vogt construction of $\mathcal{P}$ should entrywise be of the form

$$W(\mathcal{G}, J, \mathcal{P})(\mathcal{G}) = \left( \bigsqcup G J[G] \otimes \mathcal{P}[G] \right) / \sim$$

in which the commutative segment $J$ decorates the ordinary internal edges in $G$. For topological colored operads using unital trees and with $J = [0, 1]$, this quotient of a coproduct is actually the original definition of the Boardman-Vogt construction in [BV72] III.1.

The categorical concept of a coend provides a nice way of packaging such a quotient of a coproduct. So our Boardman-Vogt construction is actually defined entrywise as a coend

$$W(\mathcal{G}, J, \mathcal{P})(\mathcal{G}) = \int \mathcal{G}(\mathcal{G}) J[G] \otimes \mathcal{P}[G]$$

indexed by some category $\mathcal{G}(\mathcal{G})$, called a substitution category. The substitution category is the most natural category that can be associated to a pasting scheme. It has graphs with the indicated profile as objects and graph substitution as morphisms. It is a categorical device for keeping track of the desired relations $\sim$ in the previous paragraph. In the above coend, the functor $\mathcal{P}$ is induced by the $\mathcal{G}$-prop structure of $\mathcal{P}$. The contravariant functor $J$ is induced by the commutative segment structure of $J$.

Our Boardman-Vogt construction of a generalized prop is defined entrywise in one step as a coend indexed by a substitution category. As we explained above, this is very close to the original approach in [BV72]. On the other hand, conceptually our construction is quite different from the one in [BM06]. There the Boardman-Vogt construction of an operad was defined inductively in infinitely many steps as a sequential colimit of pushouts. In Chapter 6 we will, in fact, obtain such a filtration of our Boardman-Vogt construction simply by considering coends indexed by suitably smaller substitution categories. An advantage of our one-step coend definition is that many properties of the Boardman-Vogt construction now follow naturally from the universal properties of coends. As a consequence, many of our proofs have the same simple categorical flavor.

Each commutative segment $J$ is equipped with a counit $\epsilon : J \longrightarrow 1$, where $1$ is the monoidal unit in the ambient symmetric monoidal category. Using the counit
and the $\mathcal{G}$-prop structure maps of $P$, we obtain an augmentation

$$\eta: W(\mathcal{G}, J, P) \longrightarrow P,$$

with respect to which we regard the Boardman-Vogt construction as a resolution of $P$. To truly be deserving of the name resolution, in Chapter 7 we will prove in nice cases that the augmentation $\eta$ is a weak equivalence with $W(\mathcal{G}, J, P)$ a cofibrant $\mathcal{G}$-prop, so our Boardman-Vogt construction becomes a cofibrant resolution of $P$.

To show that the augmentation $\eta$ is a weak equivalence, we will consider the entrywise factorization

$$\text{Id} \xrightarrow{\omega} W(\mathcal{G}, J, P) \xrightarrow{\eta} P$$

in which the first map $\omega$ sends each entry of $P$ to the corresponding $P$-decorated corolla. By the 2-out-of-3 property of weak equivalences, it is enough to show that the map $\omega$ is entrywise a weak equivalence. This is accomplished in Theorem 7.2.17 via a homotopical analysis of the filtration on $W(\mathcal{G}, J, P)$.

To show that $W(\mathcal{G}, J, P)$ is a cofibrant $\mathcal{G}$-prop, in Section 5.5 we will define a refinement of the filtration on $W(\mathcal{G}, J, P)$. Most of Chapter 6 is a careful analysis of this refined filtration. This is used in Section 7.3 to establish the cofibrancy of $W(\mathcal{G}, J, P)$.

### Chapter Summaries

This monograph consists of two parts. Part 1 contains the categorical construction of the Boardman-Vogt resolution of a generalized prop and its main homotopical properties. Part 2 contains applications of the results in Part 1 and several constructions related to the Boardman-Vogt resolution of a generalized prop. We now provide a brief summary of each chapter.

Our Boardman-Vogt construction uses the machinery of pasting schemes from our previous work $\cite{YJ15}$. To make this book self contained, Part 1 begins with a preliminary Chapter 1 in which we recall the relevant concepts of graphs, graph substitution, pasting schemes, generalized props, and some important examples. For the reader’s convenience, we will also recall the definitions of a (symmetric) monoidal category, a monoidal functor, a monad, and an algebra over a monad.

In Chapter 2 we recall the original Boardman-Vogt construction of topological colored operads from $\cite{BV72}$. Although this material is not needed in later chapters, it provides historical context and motivation for some later constructions. Except for the notion of a unital tree, this chapter is independent of Chapter 1 and can be read first.

The Boardman-Vogt construction of a generalized prop $P$ associated to a pasting scheme $\mathcal{G}$ and commutative segment $J$ is defined in Chapter 3. This chapter begins with a discussion of the substitution categories $\mathcal{G}(J^2)$ and commutative segments, followed by some categorical properties of coends. Once the Boardman-Vogt construction is defined, we establish its $\mathcal{G}$-prop structure. In particular, the proof of Lemma 3.5.8 that the $\mathcal{G}$-prop structure map on the Boardman-Vogt construction is well-defined illustrates one advantage of our coend approach. The proof boils down to checking that some diagram is commutative, which in turn reduces to the universal property of the coend that defines the Boardman-Vogt construction. Many of our later proofs have the same pattern.
Further categorical properties of the Boardman-Vogt construction are established in Chapter 4. First we define the augmentation $\eta$ mentioned above. Then we show that the augmentation provides a natural factorization

$$
\xymatrix{
F G UP \ar[r]^\delta \ar@/^/[r]^{\text{counit}} & W(G, J, P) \ar[r]^\eta & P
}
$$

of the counit of a $G$-prop $P$ associated to the pair $G_0 \leq G$ of pasting schemes. The sub-pasting scheme $G_0$ contains the graphs in $G$ that have no ordinary internal edges. Roughly speaking the map $\delta$ is defined by regarding each $P$-decorated graph $P[G]$ as having a fixed length of 1 in each ordinary internal edge. The second half of this chapter contains various naturality properties of the Boardman-Vogt construction with respect to its three inputs $G$, $J$, and $P$ and the ambient symmetric monoidal category.

In Chapter 5 we define a filtration of the Boardman-Vogt construction

$$
W(G, J, P)_0 \longrightarrow W(G, J, P)_1 \longrightarrow W(G, J, P)_2 \longrightarrow \cdots \longrightarrow W(G, J, P)
$$

by taking the same coend over smaller substitution categories corresponding to graphs with at most a certain number of ordinary internal edges. An important observation regarding this filtration is Theorem 5.4.7. It says that each map in this filtration in each entry is a specific pushout, at least when the pasting scheme is connected. These pushouts are crucial in understanding homotopical properties of the Boardman-Vogt construction.

In Chapter 6 we reinterpret a map out of the Boardman-Vogt construction as a compatible family of maps out of the filtration strata. Then we study the question of extending a map from one stratum to the next. This analysis is needed later to show that in nice cases $W(G, J, P)$ is a cofibrant $G$-prop. Up to this point, the ambient category is a cocomplete symmetric monoidal category whose monoidal product commutes with colimits on both sides.

In Chapter 7 we assume the ambient category is a cofibrantly generated monoidal model category and $G$ is a connected pasting scheme. We show that, for a nice enough $G$-prop $P$ under the right homotopical conditions in the ambient category, the augmentation $\eta$ provides a cofibrant resolution of $P$. After that we study cofibrancy properties of the Boardman-Vogt construction with respect to changing $J$ and the $G$-prop. This concludes Part 1.

Part 2 begins with Chapter 8, which contains applications of the Boardman-Vogt resolution of a generalized prop. We will consider cofibrant resolutions of differential graded wheeled operads and of graphically generated generalized props, rectification, and homotopy invariant properties of homotopy bialgebras, homotopy operad algebras, and homotopy coherent diagrams.

In addition to the Boardman-Vogt resolution, one may also consider the bar resolution of a $G$-prop, which is by definition a simplicial $G$-prop. In Chapter 9 we observe that, for a shrinkable pasting scheme, the bar resolution corresponding to the pair $G_0 \leq G$ of pasting schemes is a special case of the Boardman-Vogt resolution in simplicial objects in the ambient category. Roughly speaking, a pasting scheme is shrinkable if it is connected and is closed under shrinking arbitrary ordinary internal edges. We also show that an appropriate realization of the bar resolution of a $G$-prop is a special case of the Boardman-Vogt construction.
In Chapter 10 we define a relative version of the Boardman-Vogt construction that applies to a map \( g : P \to Q \) of \( \mathcal{G} \)-props. Each entry of the relative \( W \)-construction of \( g \), denoted by \( W(\mathcal{G}, J, g) \), is defined as a colimit similar to \( W(\mathcal{G}, J, Q) \), but there is more identification corresponding to \( g \). We show that it fits inside a diagram

\[
\begin{array}{ccc}
W(\mathcal{G}, J, P) & \xrightarrow{g_*} & W(\mathcal{G}, J, Q) \\
\eta^P & & \eta^Q \\
P & \xrightarrow{g_0} & W(\mathcal{G}, J, g) \\
\downarrow g & & \downarrow \eta^g \\
Q & & \\
\end{array}
\]

of \( \mathcal{G} \)-props, in which the square is a pushout. Then we show that the relative \( W \)-construction has nice categorical and homotopical properties. In particular, in nice cases the factorization \( g = \eta^g g_0 \) involves a cofibration \( g_0 \) of \( \mathcal{G} \)-props followed by a weak equivalence \( \eta^g \). When restricted to a map between operads, our relative \( W \)-construction is different from the one in \([BM06]\); see Remark 10.2.13.

In Chapter 11 we define a relative version of the bar resolution in Chapter 9 that applies to a map of \( \mathcal{G} \)-props. We observe that, for a shrinkable pasting scheme, the relative bar resolution is a special case of the relative Boardman-Vogt construction in Chapter 10.

In Chapter 12 we study the Boardman-Vogt resolution of colored cyclic operads. Roughly speaking a cyclic operad is an operad in which there is no distinction between inputs and outputs. As applications, we obtain specific cofibrant resolutions of the cyclic operads \( \overline{\mathcal{M}} \) of Deligne-Grothendieck-Knudsen moduli spaces of stable curves with marked points, \( \mathcal{M} \) of moduli spaces of Riemann spheres with holes, a colored version of \( \overline{\mathcal{M}} \), and chain cyclic operads. Along the way, although not needed for the Boardman-Vogt resolution, we also study the coherence of colored cyclic operads.

In Chapter 13 we study the Boardman-Vogt resolution and coherence of colored modular operads. Roughly speaking a modular operad is a non-unital wheeled properad in which there is no distinction between inputs and outputs. Furthermore, the underlying object satisfies some stability conditions. As applications, we obtain specific cofibrant resolutions of the modular operads \( \overline{\mathcal{M}} \), \( \mathcal{M} \), a colored version of \( \overline{\mathcal{M}} \), and chain modular operads. This concludes Part 2.

**Chapter Interdependence.** As we mentioned above, except for the definition of a unital tree, Chapter 2 is independent of the rest of this book. The
The interdependence of the other chapters are essentially as follows:

The exceptions are Corollary 9.6.5 and Section 10.4, which use results from Chapter 7. For example, if one is particularly interested in the relative Boardman-Vogt construction, then one can read Chapters 1, 3, and 10. For the homotopical properties in Section 10.4 one can refer back to Chapter 7 as necessary.

Prerequisites. With the preliminary Chapter 1 and Chapter 2, we do not assume the reader knows about generalized props and the Boardman-Vogt construction of operads. For the homotopical results in Chapter 7, Chapter 8 and a few other places where Chapter 7 is referred, the reader is assumed to know the basics of cofibrantly generated monoidal model categories. The model category references listed in Section 7.1 provide enough background. The rest of this monograph is almost entirely categorical in nature. The reader is only assumed to be familiar with very basic concepts of category theory, including adjoint functors, coends, and colimits, the definitions of which can be found in [Mac98].

The presentation throughout this monograph is roughly at the second to third year graduate level. There are more details and many more pictures and examples in this monograph than in a journal article. Sprinkled throughout this monograph are paragraphs named Motivation, each of which conveys the intuition in an upcoming definition or result. Therefore, this monograph is suitable for graduate students as well as researchers.

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Part 1

Boardman-Vogt Resolutions
CHAPTER 1

Graphs, Pasting Schemes, and Generalized Props

The purpose of this chapter is to recall the concepts of graphs, graph substitution, pasting schemes, and generalized props from our previous work [YJ15], where the reader is referred for more detailed discussion and proofs. The following material is adapted from [YJ15] Part 1 and Chapter 10.

Generalized props, cyclic operads, and modular operads are algebras over monads parametrized by certain types of graphs. We first recall these graphs in Section 1.1 and Section 1.2 and the fundamental operation of graph substitution, which parametrizes the monadic multiplication, in Section 1.3. Pasting schemes along with some important examples are discussed in Section 1.4. In Section 1.5 we recall the definitions of a (symmetric) monoidal category, a monoidal functor, a monad, and an algebra over a monad. The definition of a generalized prop is given in Section 1.6. In Section 1.7 we discuss the free-forgetful adjunction associated to an inclusion of pasting schemes.

1.1. Graphs

The wheeled graphs that define generalized props are the following kind of graphs with some extra structure. The graphs below will also be used for colored cyclic operads and colored modular operads. Roughly speaking, a graph has vertices, edges between them, and legs that are attached to a vertex only on one side. It is also allowed to have exceptional edges and exceptional loops that are not attached to any vertex. In the following definition, the involution \( \iota_G \) is used to create internal edges (i.e., those among vertices and exceptional loops) and legs, and the involution \( \pi_G \) is used to create exceptional edges from exceptional legs. Recall that an involution is a self-map \( \tau \) such that \( \tau^2 = \text{Id} \); it is free if it has no fixed points.

**Definition 1.1.1.** Fix an infinite set \( \mathcal{F} \) once and for all. A graph is a tuple

\[
G = (\text{Flag}(G), \lambda_G, \iota_G, \pi_G)
\]

consisting of:

- a finite set \( \text{Flag}(G) \subset \mathcal{F} \) of flags;
- a partition \( \lambda_G \) of \( \text{Flag}(G) \) into finitely many possibly empty subsets, called cells, together with a distinguished cell \( G_0 \), called the exceptional cell;
- an involution \( \iota_G \) on \( \text{Flag}(G) \) such that \( \iota_G(G_0) = G_0 \);
- a free involution \( \pi_G \) on the set of \( \iota_G \)-fixed points in \( G_0 \).

An isomorphism of graphs is a bijection on flags that preserves the partition and both involutions. For graphs with any further structure as we will introduce later, an isomorphism is required to preserve that structure as well.

**Definition 1.1.2.** Suppose \( G \) is a graph.
• Flags not in the exceptional cell $G_0$ are called ordinary flags. Flags in $G_0$ are called exceptional flags.

• $G$ is said to be an ordinary graph if the exceptional cell is empty.

• A vertex is a cell that is not the exceptional cell. A flag in a vertex $v$ is said to be adjacent to $v$. An isolated vertex is a vertex that is empty. The cardinality of a vertex $v$ is denoted by $|v|$. The set of vertices is denoted by $Vt(G)$.

• The fixed points of $\iota_G$ are called legs. The set of legs is denoted by $\text{Leg}(G)$. A leg in a vertex is called an ordinary leg. A leg in the exceptional cell is called an exceptional leg.

• The orbits of $\pi_G$ and of $\iota_G$ away from its fixed points in $G_0$ are called edges. The set of edges is denoted by $\text{Ed}(G)$.

• The non-trivial orbits of $\iota_G$ are called internal edges. The non-trivial orbits of $\iota_G$ within the vertices are called ordinary internal edges. Those within the exceptional cell are called exceptional loops and denoted by $\bigcirc$.

• The orbits of $\pi_G$ are called exceptional edges and denoted by $|\cdot|$.

• If $f = \{f_+\}$ is an ordinary internal edge with $f_+ \in u$ and $f_- \in v$, where $u = v$ is allowed, then we say that $f$ is adjacent to $u$ and $v$.

Some basic examples of graphs follow.

**Example 1.1.3.** Suppose $G^i$ is a graph with exceptional cell $G^i_0$ for $1 \leq i \leq n$. Their disjoint union $G$ is the graph with

$$\text{Flag}(G) = \prod_{i=1}^{n} \text{Flag}(G^i), \quad G_0 = \prod_{i=1}^{n} G^i_0,$$

and the induced partition and involutions.

**Example 1.1.4.** The empty graph $\emptyset$ has an empty set of flags, hence an empty exceptional cell, and no vertices.

**Example 1.1.5.** The graph with an empty set of flags, hence an empty exceptional cell, and a single empty vertex is an isolated vertex.

**Example 1.1.6.** The graph with no vertices and with only two exceptional legs $f_{\pm}$, which must be paired by the involution $\pi$, is the exceptional edge $|\cdot|$.

**Example 1.1.7.** The graph with no vertices and with only two exceptional flags $e_{\pm}$ paired by $\iota$ is the exceptional loop $\bigcirc$.

**Example 1.1.8.** For $n \geq 0$ the $n$-corolla, denoted $C_n$, is the ordinary graph whose only flags are $n$ ordinary legs $l_1, \ldots, l_n$ in the same vertex $v$, in which there are no other vertices. For example, the 5-corolla is depicted as

```
      l_5
     /  \
   l_1  \\
   /    \
  l_2  \\
 /      \
l_3      l_4
```

such that a vertex is depicted as a disk, not to be confused with an exceptional loop. A leg is depicted as a line extended from its vertex.

**Example 1.1.9.** The ordinary graph depicted as
1.1. GRAPHS

has:

- 13 flags \{l_1, l_2, l_3, e, f, g, h, i, k, l, m, n, o, p\};
- three vertices \(u = \{l_1, e, f, g, h, i\}\), \(v = \{l_2, l_3, f, g, h, i\}\), and \(w = \{h, i, l\}\);
- three ordinary legs \(\{l, l_2, l_3\}\);
- five ordinary internal edges \(a = \{a, e, f, g, h, i\}\) for \(a \in \{e, f, g, h, i\}\).

We will use the following concepts to define (simple) connectivity.

**Definition 1.1.10.** Suppose \(G\) is a graph.

1. A path of length \(r \geq 0\) is a pair \(P = (\{e_i\}_{i=1}^{r}, \{v_i\}_{i=0}^{r})\) in which:
   - the \(v_i\)'s are distinct vertices, except possibly for \(v_0\) and \(v_r\);
   - each \(e_i\) is an ordinary internal edge adjacent to both \(v_{i-1}\) and \(v_i\);

2. A cycle is a path of length \(r \geq 1\) with \(v_0 = v_r\).

**Example 1.1.11.** If \(v\) is a vertex in a graph, then \((\emptyset, \{v\})\) is a path of length 0.

**Example 1.1.12.** Consider the ordinary graph in Example 1.1.9

1. \((\{g, i\}, \{u, v, w\})\) is a path of length 2.
2. \((\{e\}, \{u, v\})\) is a cycle of length 1.
3. \((\{f, g\}, \{v, u, v\})\) is a cycle of length 2.
4. \((\{f, h, i\}, \{v, u, w, v\})\) is a cycle of length 3.

**Definition 1.1.13.** A non-empty graph \(G\) is connected if it satisfies one of the following two conditions:

1. It an isolated vertex (Example 1.1.5), the exceptional edge (Example 1.1.6), or the exceptional loop (Example 1.1.7).
2. It is an ordinary graph that has no isolated vertices. Moreover, for each pair of distinct flags \(\{f_1, f_2\}\), there exists a path \(P = (\{e_i\}, \{v_i\})\) such that \(f_1\) is adjacent to some \(v_i\) and that \(f_2\) is adjacent to some \(v_j\).

**Example 1.1.14.** The \(n\)-corolla in Example 1.1.8 and the graph in Example 1.1.9 are both connected.

**Example 1.1.15.** If \(G^1\) and \(G^2\) are non-empty graphs, then their disjoint union (Example 1.1.3) is not connected.

**Definition 1.1.16.** A connected graph is simply-connected if it is not the exceptional loop and contains no cycles.
EXAMPLE 1.1.17. An isolated vertex, the exceptional edge, and the \( n \)-corolla are simply-connected.

EXAMPLE 1.1.18. The graph in Example 1.1.9 is not simply-connected. On the other hand, if we remove the edges \( e \), \( f \), and \( i \) from it, then the resulting graph

\[
\begin{array}{ccc}
& u & h \\
& \downarrow & \downarrow \\
& w & & v \\
& & l_1 & l_3 \\
& & & l_2 \\
\end{array}
\]

is simply-connected.

### 1.2. Structures on Graphs

We will usually consider graphs whose edges are equipped with colors in the following sense. Fix a non-empty set \( C \) once and for all, whose elements will be called colors.

**Definition 1.2.1.** A \( C \)-coloring for a graph \( G \) is a function

\[ \kappa : \text{Flag}(G) \to C \]

that is constant on the orbits of both involutions \( \iota_G \) and \( \pi_G \).

This definition amounts to saying that each edge is assigned a color.

**Example 1.2.2.** Suppose \( G \) is a graph.

1. There is a coloring on \( G \) with all the flags given the same color. In other words, each graph can be regarded as a one-colored graph.
2. There is an \( \text{Ed}(G) \)-coloring on \( G \) in which each edge is colored by itself. In other words, each edge is assigned a unique color.

For cyclic operads and modular operads, the corresponding graphs are not directed. On the other hand, for generalized props, such as operads and wheeled props, we need to work with directed graphs in the following sense.

**Definition 1.2.3.** A direction for a graph \( G \) is a function

\[ \delta : \text{Flag}(G) \to \{1, -1\} \]

such that:

- If \((f, \iota_G(f))\) is an internal edge, then \(\delta(\iota_G(f)) = -\delta(f)\).
- If \((f, \pi_G(f))\) is an exceptional edge, then \(\delta(\pi_G(f)) = -\delta(f)\).

**Definition 1.2.4.** Suppose \( G \) is a graph equipped with a direction \( \delta \).

- A leg \( f \) with \( \delta(f) = 1 \) is an input of \( G \).
- A leg \( f \) with \( \delta(f) = -1 \) is an output of \( G \).
- If \( v \) is a vertex with \( f \in v \) and \( \delta(f) = 1 \), then \( f \) is an input of \( v \).
- If \( v \) is a vertex with \( f \in v \) and \( \delta(f) = -1 \), then \( f \) is an output of \( v \).
- For \( z \in \{G \cup \text{Vt}(G)\} \), the set of inputs of \( z \) is denoted by \( \text{in}(z) \), and the set of outputs of \( z \) is denoted by \( \text{out}(z) \).
- We regard an internal edge as oriented from the flag with \( \delta = -1 \) to the flag with \( \delta = 1 \).
1.2. STRUCTURES ON GRAPHS

 For an ordinary internal edge $e = \{ e_\pm \}$ with $\delta(e_\pm) = \pm 1$, the vertex containing $e_-$ (resp., $e_+$) is the initial vertex (resp., terminal vertex) of $e$.

**Example 1.2.5.** The exceptional edge in Example 1.1.6 can be equipped with a direction in which one flag is assigned $\delta = 1$ (the input) and the other flag is assigned $\delta = -1$ (the output). We denote it by $\uparrow$, where the arrow half is the output. If, furthermore, it has color $c \in \mathcal{C}$, then we write $\uparrow_c$.

**Example 1.2.6.** The exceptional loop in Example 1.1.7 can be equipped with a direction in which one flag is assigned $\delta = 1$ and the other flag is assigned $\delta = -1$. We denote it by $\downarrow_c$. If, furthermore, it has color $c \in \mathcal{C}$, then we write $\downarrow_c$.

**Example 1.2.7.** For the 5-corolla in Example 1.1.8, if we define $\delta(l_1) = \delta(l_2) = \delta(l_3) = -1$ and $\delta(l_4) = \delta(l_5) = 1$, then we have the corolla $C_{(2;3)}$

![Diagram of a 5-corolla]

with two inputs $\{l_3, l_4\}$ and three outputs $\{l_1, l_2, l_5\}$.

**Example 1.2.8.** One way to provide the graph $G$ in Example 1.1.9 with a direction is to define $\delta(l_1) = \delta(l_2) = -1$, $\delta(l_3) = 1$, and $\delta(a_\pm) = \pm 1$ for $a \in \{e, f, g, h, i\}$. With this direction this graph may be depicted as

![Diagram of a graph with directed edges]

with $\text{in}(G) = \{l_3\}$ and $\text{out}(G) = \{l_1, l_2\}$.

**Definition 1.2.9.** Suppose $G$ is a graph.

1. An ordering at a vertex $v$ is a bijection $\ell_v : \{1, \ldots, |v|\} \rightarrow v$.

2. An ordering of $G$ is a bijection $\ell_G : \{1, \ldots, |\text{Leg}(G)|\} \rightarrow \text{Leg}(G)$.

A listing of $G$ is a choice of an ordering for each $z \in \{G\} \sqcup \text{Vt}(G)$. Given a listing, we will regard $\text{Leg}(G)$ and each vertex $v$ as ordered sets.

**Definition 1.2.10.** Suppose $G$ is a graph with a direction.

1. An ordering at a vertex $v$ is a pair of bijections $\ell_v = (\ell_v^\text{in}, \ell_v^\text{out}) : (\{1, \ldots, |\text{in}(v)|\}; \{1, \ldots, |\text{out}(v)|\}) \rightarrow (\text{in}(v), \text{out}(v))$. 


(2) An ordering of $G$ is a pair of bijections
\[
\ell_G = (\ell_G^{\text{in}}; \ell_G^{\text{out}}) : \{1, \ldots, |\text{in}(G)|\} \cup \{1, \ldots, |\text{out}(G)|\} \rightarrow (\text{in}(G), \text{out}(G)).
\]
A listing of $G$ is a choice of an ordering for each $z \in \{G\} \cup \text{Vt}(G)$. Given a listing, we will regard each $(\text{in}(z); \text{out}(z))$ as a pair of ordered sets.

Definition 1.2.11. For a given non-empty set $\mathcal{C}$, a $\mathcal{C}$-profile is a finite, possibly empty, sequence of colors.

1. The set of $\mathcal{C}$-profiles is denoted by $\text{Prof}(\mathcal{C})$.
2. If $\mathcal{C} = (c_1, \ldots, c_m)$ is a $\mathcal{C}$-profile, then $|\mathcal{C}| = m$ is called its length. The empty profile is denoted by $\varnothing$.
3. A pair $(\mathcal{C}; d) \in \text{Prof}(\mathcal{C})^*\!\!\setminus\!\!, \delta$ is also written vertically as $(\mathcal{C})$.
4. If $\mathcal{C}$ is the one-point set, then we will denote the $\{\ast\}$-profile of length $m$ by $m$.

Example 1.2.12. If $\mathcal{C}$ is the one-point set, then $\text{Prof}((\ast)) = \mathbb{N} = \{0, 1, 2, \ldots\}$.

Definition 1.2.13. Suppose $G$ is a graph with a $\mathcal{C}$-coloring $\kappa$ and a listing $\ell$.

1. For each vertex $v$, the profile of $v$ is
\[
\text{Prof}(v) = (\kappa(\ell(v)(1)), \ldots, \kappa(\ell(v)(|v|))) \in \text{Prof}(\mathcal{C}),
\]
which we sometimes abbreviate to just $v$.
2. The profile of $G$ is
\[
\text{Prof}(G) = (\kappa(\ell_G(1)), \ldots, \kappa(\ell_G(|\text{Leg}(G)|))) \in \text{Prof}(\mathcal{C}).
\]

We make similar definitions if, furthermore, $G$ is equipped with a direction. In this case, for $z \in \{G\} \cup \text{Vt}(G)$, the profile of $z$ is a pair
\[
\text{Prof}(z) = (\text{in}(z); \text{out}(z)) \in \text{Prof}(\mathcal{C})^{*\!\!\setminus\!\!, \delta},
\]
where $\kappa$ has been suppressed from the notation. We call $\text{in}(z)$ the input profile and $\text{out}(z)$ the output profile of $z$.

Definition 1.2.14. A $\mathcal{C}$-colored wheeled graph is a tuple
\[
(G, \kappa, \delta, \ell)
\]
with $G$ a graph, $\kappa$ a $\mathcal{C}$-coloring, $\delta$ a direction, and $\ell$ a listing. The groupoid of $\mathcal{C}$-colored wheeled graphs is denoted by $\text{Gr}^\mathcal{C}$.

Example 1.2.15. The empty graph in Example 1.1.4 and the isolated vertex in Example 1.1.5 are both wheeled graphs with profiles $(\varnothing; \varnothing)$.

Example 1.2.16. The exceptional edge $\uparrow^c$ in Example 1.2.5 has a unique listing and profile $(c; c) \in \text{Prof}(\mathcal{C})^\{2\}$. The $\uparrow^c$-corolla $C_{(c; c)}$ is the ordinary connected graph with a $\mathcal{C}$-coloring $\kappa$, a direction $\delta$, and a listing $\ell$ defined as follows.
• Flag\(\{C(\underline{c}, \underline{d})\} = \{i_1, \ldots, i_m, o_1, \ldots, o_n\}\), all of which are legs at a unique vertex \(v\).
• \(\kappa(i_p) = c_p\) and \(\kappa(o_q) = d_q\).
• \(\delta(i_p) = 1\) and \(\delta(o_q) = -1\).
• \(\ell^\text{in}_z(p) = i_p\) for \(z \in \{v, C(\underline{c}, \underline{d})\}\) and \(1 \leq p \leq m\).
• \(\ell^\text{out}_z(q) = o_q\) for \(z \in \{v, C(\underline{c}, \underline{d})\}\) and \(1 \leq q \leq n\).

Note that
\[
\text{Prof}(v) = \text{Prof}(\{C(\underline{c}, \underline{d})\}) = (\underline{c}; \underline{d}) \in \text{Prof}(\mathcal{C})^2.
\]

We depict the \((\underline{c}, \underline{d})\)-corolla as
\[
\begin{array}{c}
\circ \\
\downarrow & \downarrow \\
c_1 & \cdots & c_m
\end{array}
\begin{array}{c}
\downarrow & \downarrow \\
d_1 & \cdots & d_n
\end{array}
\]
in which the flags are drawn from left to right according to their ordering.

**Example 1.2.19.** With the same setting as in Example 1.2.18, suppose given permutations \(\sigma \in \Sigma_n\) and \(\tau \in \Sigma_m\). The permuted corolla
\[
\sigma C(\underline{c}, \underline{d}) \tau
\]
is defined just like the corolla \(C(\underline{c}, \underline{d})\), except for the ordering of the whole graph:
\[
\begin{align*}
\ell^\text{in}_{\sigma C(\underline{c}, \underline{d}) \tau}(p) &= i_{\tau(p)} \quad \text{for} \quad 1 \leq p \leq m, \\
\ell^\text{out}_{\sigma C(\underline{c}, \underline{d}) \tau}(q) &= o_{\sigma^{-1}(q)} \quad \text{for} \quad 1 \leq q \leq n.
\end{align*}
\]

Note that
\[
\text{Prof}(v) = (\underline{c}; \underline{d}) \quad \text{and} \quad \text{Prof}(\sigma C(\underline{c}, \underline{d}) \tau) = (\underline{c}; \underline{d}) \in \text{Prof}(\mathcal{C})^2.
\]

For example, if \(c = (c_1, c_2), d = (d_1, d_2, d_3), \tau = (1 \ 2) \in \Sigma_2,\) and \(\sigma = (1 \ 3 \ 2) \in \Sigma_3,\) then we may visualize the permuted corolla \(\sigma C(\underline{c}, \underline{d}) \tau\) as:
\[
\begin{array}{c}
\circ \\
\downarrow & \downarrow \\
c_1 & \downarrow \\
\downarrow & \downarrow \\
d_1 & \cdots & d_3
\end{array}
\begin{array}{c}
\downarrow & \downarrow \\
d_2 & \cdots & d_3
\end{array}
\]

Permuted corollas will provide generalized props with their equivariant structure.

**Example 1.2.20.** In the corolla \(C(\underline{c}, \underline{d})\) in Example 1.2.18, suppose \(c_k = d_l\) for some \(1 \leq k \leq m = |\underline{c}|\) and \(1 \leq l \leq n = |\underline{d}|\). There is a contracted corolla
\[
\xi^l_k C(\underline{c}, \underline{d})
\]
that is defined just like \(C(\underline{c}, \underline{d})\) except that:
• The flags \(i_k\) and \(o_l\) form an internal edge, which is therefore a loop at the unique vertex \(v\).
• The leg ordering is given by
\[
\begin{align*}
\ell^\text{in}_{\xi^l_k C(\underline{c}, \underline{d})}(p) &= \begin{cases} i_p & \text{if } 1 \leq p < k, \\
n_i & \text{if } k \leq p \leq m - 1, \end{cases} \\
\ell^\text{out}_{\xi^l_k C(\underline{c}, \underline{d})}(q) &= \begin{cases} o_q & \text{if } 1 \leq q < l, \\
n_o & \text{if } l \leq q \leq n - 1. \end{cases}
\end{align*}
\]
The contracted corolla may be depicted as:

![Graph Diagram]

Contracted corollas correspond to the contractions in wheeled props, wheeled pr-operads, and wheeled operads.

### 1.3. Graph Substitution

Generalized props, cyclic operads, and modular operads will be defined as algebras over monads parametrized by suitable families of graphs. The monadic multiplication corresponds to graph substitution, which we define next.

**Convention 1.3.1.** Unless otherwise specified, each graph will be equipped with a $C$-coloring and a listing. So a wheeled graph is a graph with a direction.

**Motivation 1.3.2.** Suppose given a (wheeled) graph $G$ and for each vertex $v$ in $G$, a (wheeled) graph $H_v$ whose profile is the same as $v$. The graph substitution $G(H_v)_{v \in G}$ that we will define is intuitively obtained by cutting a small hole around each vertex $v$ and patching in a copy of $H_v$.

![Graph Substitution Diagram]

(1.3.3)
The resulting graph has the same profile as $G$.

**Definition 1.3.4.** Suppose given an ordinary (wheeled) graph $G$ and, for each vertex $v \in G$, an ordinary (wheeled) graph $H_v$ whose profile is the same as that of $v$. The **graph substitution**

$$G(H_v)_{v \in G}$$

is the ordinary (wheeled) graph whose set of flags is

$$\text{Flag}(G(H_v)) = \bigsqcup_{v \in G} \text{Flag}(H_v)$$

with the induced partition $\lambda_{G(H_v)} = \bigsqcup \lambda_{H_v}$. The involution $\iota_{G(H_v)}$ on $\text{Flag}(G(H_v))$ is given by:

- $\iota_{H_v}$ for non-leg flags in $H_v$;
- $\iota_{G}$ for legs in the $H_v$’s after identifying the $i$th leg in each $H_v$ with the $i$th flag in $v$. 
1.3. GRAPH SUBSTITUTION

The \( C \)-coloring, the ordering at each vertex, and the direction (for wheeled graphs) are induced by those of the \( H_v \)'s. The ordering on the set of legs is induced by that of \( G \). We say \( H_v \) is substituted into \( v \).

If \( G \) and/or some of the \( H_v \)'s are not ordinary, then the graph substitution \( G(H_v) \) can still be defined, but the definition is more complicated because it involves a carefully defined quotient. The reader is referred to \([YJ15]\) Chapter 5 for details. Below we will recall some basic properties of graph substitution. Each statement has a version for graphs and a version for wheeled graphs. For simplicity we will only state the wheeled version here.

**Theorem 1.3.5.** Consider the setting of Def. 1.3.4.

1. The graph substitution \( G(H_v) \) has the same profile as \( G \).
2. Exceptional flags in \( G \) remain exceptional flags in the graph substitution.
3. The vertices in the graph substitution come from the inside graphs: \( \bigcup_{v \in G} \text{Vt}(H_v) = \bigcup_{v \in G} \text{Vt}(H_v) \).
4. Graph substitution is associative in the following sense. Suppose given \( I_u \) with the same profile as \( u \) for each vertex \( u \) in each \( H_v \). Then there is a canonical isomorphism
   \[
   \left[ G(H_v) \right] (I_u) \cong G \left[ H_v(I_u) \right].
   \]
5. Graph substitution is unital in the sense that there are canonical isomorphisms:
   - \( G(C_v) \cong G \), where \( C_v \) is the \( \text{Prof}(v) \)-corolla in Example 1.2.18
   - \( C(G) \cong G \), where \( C \) is the \( \text{Prof}(G) \)-corolla.
6. If \( \sigma C \tau \) is a permuted corolla as in Example 1.2.19, then the graph substitution \( (\sigma C \tau)(G) \) is canonically isomorphic to \( G \) except that its profile is permuted as
   \[
   \text{Prof}((\sigma C \tau)(G)) = (\text{in}(G) \tau; \text{out}(G)).
   \]
7. If \( \sigma_v C_v \tau_v \) is a permuted corolla with the same profile as \( v \), then the graph substitution \( G(\sigma_v C_v \tau_v) \) is canonically isomorphic to \( G \) except that its vertex ordering is permuted as
   \[
   \text{Prof}(v') = (\text{in}(v) \tau_v^{-1}; \sigma_v^{-1} \text{out}(v)).
   \]

Different types of generalized props correspond to different sub-classes of wheeled graphs. We now define some of the commonly used classes of wheeled graphs.

**Definition 1.3.6.** A directed path/cycle is a path/cycle as in Def. 1.1.10 such that each \( e_i \) has initial vertex \( v_{i-1} \) and terminal vertex \( v_i \).

1. A wheel-free graph is a wheeled graph with neither exceptional loops \( Q \) (Example 1.2.6) nor directed cycles. The groupoid of wheel-free graphs is denoted by \( \text{Gr}^1 \).
2. The groupoid of connected (Def. 1.1.13) wheeled graphs is denoted by \( \text{Gr}^1 \).
3. The groupoid of connected wheel-free graphs is denoted by \( \text{Gr}^1_1 \).
4. The groupoid of simply-connected (Def. 1.1.16) wheel-free graphs is denoted by \( \text{Gr}^1_{sc} \).
(5) A **wheeled tree** is a connected wheeled graph in which each vertex $v$ satisfies

$$|\text{out}(v)| \leq 1.$$  

The groupoid of wheeled trees is denoted by $\text{Tree}^\bigcirc$.

(6) A **unital tree** is a simply-connected wheel-free graph in which each vertex $v$ satisfies

$$|\text{out}(v)| = 1.$$  

The groupoid of unital trees is denoted by $\text{UTree}$.

(7) A **unital linear graph** is a unital tree in which each vertex $v$ satisfies

$$|\text{in}(v)| = 1.$$  

The groupoid of unital linear graphs is denoted by $\text{ULin}$.

**Proposition 1.3.7.** The groupoids $\text{Gr}^\bigcirc$, $\text{Gr}_e^\bigcirc$, $\text{Gr}^\uparrow$, $\text{Gr}^\uparrow_c$, $\text{Gr}^\uparrow_{sc}$, $\text{Tree}^\bigcirc$, $\text{UTree}$, and $\text{ULin}$ are all closed under graph substitution.

**Example 1.3.8.** The exceptional edge $\uparrow_c$ is a unital linear graph, since it is simply-connected and has no vertices to check the required condition $|\text{in}(v)| = |\text{out}(v)| = 1$.

**Example 1.3.9.** The exceptional loop $\bigcirc_e$ is a wheeled tree, since it is connected and has no vertices to check the required condition $|\text{out}(v)| \leq 1$.

**Example 1.3.10.** The permutted corolla $\sigma C_{(e,d)}^\tau$ is simply-connected. It is a wheeled tree if and only if $|d| \leq 1$. It is a unital tree if and only if $|d| = 1$. It is a unital linear graph if and only if $|c| = |d| = 1$.

**Example 1.3.11.** The contracted corolla $\xi_k C_{(e,d)}$ with $c_k = d_l$ in Example 1.2.20 is a connected wheeled graph that is not wheel-free. It is a wheeled tree if and only if $|d| = 1$, in which case it looks like:

![Diagram](image)

**Example 1.3.12.** The wheeled graph in Example 1.2.8 and $G$ in 1.3.3 are connected wheeled graphs, but they are neither wheeled trees nor wheel-free graphs.

**Example 1.3.13.** The two wheeled graphs

![Diagram](image)

are connected wheel-free graphs, but they are not simply-connected.
Example 1.3.14. The three wheeled graphs

are all wheeled trees, but they are not wheel-free graphs.

Example 1.3.15. The wheeled graph

is simply-connected, but it is not a unital tree because $|\text{out}(w)| = 2$.

1.4. Pasting Schemes

Recall that a sub-category is called replete in the larger category if it contains any object isomorphic in the larger category to an object of the sub-category. A groupoid is a category in which every morphism is an isomorphism.

Definition 1.4.1. The groupoid $\Sigma_C$ has object set $\text{Prof}(C)$ (Def. 1.2.11) and left permutations

$$\sigma : z = (c_1, \ldots, c_m) \mapsto \sigma z = (c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(m)})$$

as isomorphisms. The opposite groupoid $\Sigma_C^{\text{op}}$ has right permutations

$$z \mapsto \sigma \sigma = \sigma^{-1} z$$

as isomorphisms.

The groupoids $\Sigma_C$ and $\Sigma_C^{\text{op}}$ will be used to parametrize output and input profiles, respectively. The role of $S$ in a pasting scheme below is to control the allowable pairs of input/output profiles.

Definition 1.4.2. For a given non-empty set $\mathcal{C}$, a $\mathcal{C}$-colored pasting scheme is a pair

$$\mathcal{G} = (S, G)$$

in which

- $S$ is a replete and full sub-groupoid of $\Sigma_{\text{op}} C \times \Sigma_C$.
- $G$ is a replete and full sub-groupoid of $\text{Gr}^{\text{op}}$ that:
  - satisfies $\text{Prof}(z) \in S$ for each $G \in G$ and each $z \in \{G\} \cup \text{Vt}(G)$;
  - contains all the permuted corollas $\sigma C_{(z; d)} \tau$ for $(z; d) \in S$;
  - is closed under graph substitution.
A pasting scheme is **unital** if, furthermore, for each color \( c \in \mathcal{C} \) that appears in some element in \( S, \uparrow_{c} \in \mathcal{G} \).

**Remark 1.4.3.** Consider Def. 1.4.2

1. The above definition of a unital pasting scheme is slightly weaker than the one in \([YJ15]\) Def. 8.6, which contains an extra condition that we do not need in this work.

2. By the last two statements in Theorem 1.3.5, \( \mathcal{G} \) is closed under permutations of the ordering at each vertex and of the legs of each graph.

3. When \( S \) is understood, we will suppress it from the notation and blur the difference between \( \mathcal{G} \) and \( \mathcal{G} \). When \( \mathcal{C} \) is understood, we will omit mentioning it as well.

**Example 1.4.4.** If \( S \) is not specified, that means it is all of \( \Sigma^{\text{op}} \times \Sigma_{\mathcal{E}} \).

1. There is a unital pasting scheme \( \text{Gr}^\mathcal{Q} \) of all wheeled graphs.

2. There is a unital pasting scheme \( \text{Gr}_{\mathcal{C}}^\mathcal{Q} \) of all connected wheeled graphs.

3. There is a unital pasting scheme \( \text{Tree}^\mathcal{Q} \) of all wheeled trees, where \( S \) contains the pairs \( (c,d) \) with \( |d| \leq 1 \).

4. There is a unital pasting scheme \( \text{Gr}^\mathcal{Q}_c \) of all wheel-free graphs.

5. There is a unital pasting scheme \( \text{Gr}^\mathcal{Q}_{\mathcal{C}} \) of all simply-connected wheel-free graphs.

6. There is a unital pasting scheme \( \text{U} \text{Tree} \) of all unital trees, where \( S \) contains the pairs \( (c,d) \) with \( |d| = 1 \).

7. There is a unital pasting scheme \( \text{UL} \text{in} \) of all unital linear graphs, where \( S \) contains the pairs \( (c,d) \) with \( |d| = 1 \).

8. For each replete and full sub-groupoid \( S \) of \( \Sigma^{\text{op}} \times \Sigma_{\mathcal{E}} \), there is a pasting scheme \( \text{Cor}_{S} \) that contains only the permuted corollas \( \sigma C_{(c,d)}^\tau \) for \((c,d) \in S\).

The following concept of a pasting scheme inclusion will yield change of pasting schemes adjunctions.

**Definition 1.4.5.** Suppose \( \mathcal{G} = (S, \mathcal{G}) \) and \( \mathcal{G}' = (S', \mathcal{G}') \) are pasting schemes. An inclusion

\[ \mathcal{G} \leq \mathcal{G}' \]

is defined whenever \( S \subseteq S' \) and \( \mathcal{G} \subseteq \mathcal{G}' \) are sub-groupoids.

**Example 1.4.6.** We have the following inclusions of pasting schemes:

\[
\begin{array}{ccccccccc}
\text{U} \text{Lin} & \rightarrow & \text{U} \text{Tree} & \rightarrow & \text{Gr}^\mathcal{Q}_{\mathcal{C}} & \rightarrow & \text{Gr}^\mathcal{Q} & \rightarrow & \text{Gr}^\mathcal{Q}_c & \rightarrow & \text{Gr}^\mathcal{Q}_c \\
\uparrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\text{Tree}^\mathcal{Q} & \rightarrow & \text{Gr}^\mathcal{Q}_{\mathcal{C}} & \rightarrow & \text{Gr}^\mathcal{Q} & & & & & & \\
\end{array}
\]

For instance, the inclusion \( \text{Gr}^\mathcal{Q}_{\mathcal{C}} \leq \text{Gr}^\mathcal{Q}_{\mathcal{C}} \) will yield the free-forgetful adjunction between dioperads and properads.

**Example 1.4.7.** For each replete and full sub-groupoid \( S \) of \( \Sigma^{\text{op}} \times \Sigma_{\mathcal{E}} \), \( \text{Cor}_{S} \) is the minimal pasting scheme with that \( S \). So if \( \mathcal{G} = (S, \mathcal{G}) \), then there is a pasting
1.5. Symmetric Monoidal Categories and Monads

Here we recall some definitions regarding symmetric monoidal categories and monads. The reader is referred to [Mac98] VI, VII, and XI for details.

A monoidal category is a tuple

\[ \mathcal{M} = (\mathcal{M}, \otimes, 1, \alpha, \lambda, \rho) \]

consisting of
- a category \( \mathcal{M} \),
- a bifunctor \( \otimes: \mathcal{M} \times \mathcal{M} \to \mathcal{M} \) called the monoidal product,
- an object \( 1 \in \mathcal{M} \) called the monoidal unit, and
- three natural isomorphisms
  \[ \alpha: X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z, \]
  \[ \lambda: 1 \otimes X \cong X, \quad \text{and} \quad \rho: X \otimes 1 \cong X \]

for \( X, Y, Z \in \mathcal{M} \).

It is required that the Pentagon Diagram

\[
\begin{array}{ccc}
(W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{\alpha} & (W \otimes (X \otimes Y)) \otimes Z \\
\downarrow{\text{Id}_W \otimes \alpha} & & \downarrow{\alpha \otimes \text{Id}_Z} \\
W \otimes ((X \otimes Y) \otimes Z) & \xrightarrow{\alpha} & (W \otimes (X \otimes Y)) \otimes Z
\end{array}
\]

be commutative for all \( W, X, Y, Z \in \mathcal{M} \). Moreover, it is required that the diagram

\[
\begin{array}{ccc}
X \otimes (1 \otimes Y) & \xrightarrow{\alpha} & (X \otimes 1) \otimes Y \\
\downarrow{\text{Id}_X \otimes \lambda} & & \downarrow{\rho \otimes \text{Id}_Y} \\
X \otimes Y & \xrightarrow{\text{Id}} & X \otimes Y
\end{array}
\]

be commutative for all \( X, Y \in \mathcal{M} \) and that

\[ \lambda = \rho: 1 \otimes 1 \to 1. \]

Note that what is called a monoidal category here is sometimes called a lax monoidal category in the literature. A strict monoidal category is a monoidal category in which the structure isomorphisms \( \alpha, \lambda, \) and \( \rho \) are all identity maps. We will abbreviate a monoidal category to \( (\mathcal{M}, \otimes, 1) \) or even just \( \mathcal{M} \).

A symmetric monoidal category is a monoidal category equipped with a symmetry isomorphism

\[ \sigma = \sigma_{X,Y}: X \otimes Y \cong Y \otimes X \]
that is natural in $X, Y \in M$ such that
\[ \sigma_{Y,X} \sigma_{X,Y} = \text{Id}_{X \otimes Y}, \quad \rho = \lambda \sigma_{X,1} : X \otimes 1 \cong X, \]
and that the diagram
\[
\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{\alpha} & (X \otimes Y) \otimes Z \\
\text{Id} \otimes \sigma & \downarrow & \downarrow \sigma \\
X \otimes (Z \otimes Y) & \xrightarrow{\alpha} & (X \otimes Z) \otimes Y
\end{array}
\]
is commutative for all $X, Y, Z \in M$. A symmetric monoidal category is \textit{closed} if for each object $X$ the functor $- \otimes X$ admits a right adjoint.

\textbf{Example 1.5.1.} Here are some examples of symmetric monoidal closed categories with all small limits and colimits.

(1) The category $\text{Top}$ of compactly generated topological spaces with the cartesian product $\times$ as the monoidal product.

(2) The category $\text{SSet}$ of simplicial sets with the level-wise cartesian product $\times$ as the monoidal product.

(3) The category $\text{Ch}(k)$ of (bounded or unbounded) chain complexes of $k$-modules for a field $k$ with chain complex tensor product as the monoidal product.

(4) The category $\text{SMod}(k)$ of simplicial $k$-modules with level-wise tensor product of $k$-modules as the monoidal product.

(5) The category $\text{Cat}$ of all small categories with categorical product as the monoidal product.

(6) The category $\text{Sp}^\Sigma$ of symmetric spectra $[\text{HSS00}]$.

In a monoidal category $(M, \otimes, 1)$, a \textit{monoid} is a tuple $(X, \mu, \nu)$ consisting of
\begin{itemize}
  \item an object $X$ in $M$,
  \item a \textit{multiplication} $\mu : X \otimes X \rightarrow X$, and
  \item a \textit{unit} $\nu : 1 \rightarrow X$.
\end{itemize}
It is required that the associativity diagram
\[
\begin{array}{ccc}
X \otimes (X \otimes X) & \xrightarrow{\alpha} & (X \otimes X) \otimes X \\
\text{Id} \otimes \mu & \downarrow & \downarrow \mu \\
X \otimes X & \xrightarrow{\mu} & X
\end{array}
\]
and the unity diagram
\[
\begin{array}{ccc}
1 \otimes X & \xrightarrow{\nu \otimes \text{Id}} & X \otimes X \\
\mu \otimes \text{Id} & \downarrow & \downarrow \mu \\
X \otimes 1 & \xrightarrow{\text{Id} \otimes \nu} & X \otimes 1
\end{array}
\]
be commutative. Dually, a \textit{comonoid} is a tuple $(X, \delta, \epsilon)$ consisting of
\begin{itemize}
  \item an object $X$ in $M$,
  \item a \textit{comultiplication} $\delta : X \rightarrow X \otimes X$, and
  \item a \textit{counit} $\epsilon : X \rightarrow 1$.
\end{itemize}
It is required that the comultiplication be coassociative and counital.

A **monoidal functor**

\[
(f, f_1, f_0) : (M, \otimes_M, I_M) \to (N, \otimes_N, I_N)
\]

consists of

- a functor \( f : M \to N \),
- a natural map \( f_1 : f(X) \otimes_N f(Y) \to f(X \otimes_M Y) \) in \( N \) for \( X, Y \in M \), and
- a map \( f_0 : I_N \to f(I_M) \) in \( N \)

such that the associativity diagram

\[
\begin{array}{ccc}
\text{Id} \otimes f_1 & \xrightarrow{\alpha_N} & f_1 \otimes \text{Id} \\
\downarrow f_1 & & \downarrow f_1 \\
\alpha_M & & \\
f(X) \otimes_N f(Y \otimes_M Z) & \xrightarrow{f_1} & f(X \otimes_M (Y \otimes_M Z))
\end{array}
\]

and the unity diagrams

\[
\begin{array}{ccc}
\text{Id} \otimes f_0 & \xrightarrow{\rho_M} & f_0 \otimes \text{Id} \\
\downarrow f_1 & & \downarrow f_1 \\
\delta_M & & \\
f(X) \otimes_N I_N & \xrightarrow{f_1} & f(X \otimes_M I_M)
\end{array}
\]

\[
\begin{array}{ccc}
\delta_M & & \\
\downarrow f_1 & & \downarrow f_1 \\
I_N \otimes_N f(X) & \xrightarrow{\lambda_M} & f(I_M) \otimes_N f(X)
\end{array}
\]

are commutative for all \( X, Y, Z \in M \).

A monoidal functor is **strong** if the maps \( f_1 \) and \( f_0 \) are isomorphisms. We will abbreviate a monoidal functor to just \( f \). Note that what is called a monoidal functor here is sometimes called a **lax** monoidal functor in the literature.

By Mac Lane’s Coherence Theorem, each monoidal category is categorically equivalent to a strict monoidal category via strong monoidal functors. With this Coherence Theorem in mind, we will treat our monoidal categories as strict and omit parentheses when we take iterated tensor products.

**Example 1.5.2.** For a category \( C \), there is a functor category \( \text{Func}(C) \) whose objects are functors \( C \to C \) with natural transformations as morphisms. With composition of functors as the monoidal product and the identity functor on \( C \) as the monoidal unit, the functor category \( \text{Func}(C) \) becomes a strict monoidal category.

**Example 1.5.3.** A monoid in the monoidal category \( \text{Func}(C) \) is called a **monad** on \( C \). In other words, a monad on \( C \) is a tuple

\[
(F, \mu, \nu)
\]

in which:
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- \( F : C \to C \) is a functor.
- \( \mu : FF \to F \) is a natural transformation called the \textit{monadic multiplication}.
- \( \nu : \text{Id}_C \to F \) is a natural transformation called the \textit{monadic unit}.

It is required that the associativity and unity diagrams

\[
\begin{array}{ccc}
FF\mu & \xrightarrow{\mu F} & FF \\
\downarrow F & \mu & \downarrow F \\
F & \mu & F
\end{array}
\quad \quad
\begin{array}{ccc}
F & \xrightarrow{\mu} & FF \\
\downarrow F & \mu & \downarrow F \\
FF & \mu & FF
\end{array}
\]

be commutative. Dually, a \textit{comonad} on \( C \) is a comonoid in the monoidal category \( \text{Func}(C) \).

Suppose \((M, \otimes, \mathbb{1})\) is a monoidal category and \((F, \mu, \nu)\) is a monad on \( M \). An \textit{F-algebra} is a pair \((X, \gamma)\) consisting of

- an object \( X \in M \), and
- a structure map \( \gamma : FX \to X \).

It is required that the associativity and unity diagrams

\[
\begin{array}{ccc}
FFX\mu & \xrightarrow{\mu} & FX \\
\gamma & \downarrow F & \downarrow F \\
FX & \gamma & X
\end{array}
\quad \quad
\begin{array}{ccc}
X & \xrightarrow{\nu} & FX \\
\gamma & \downarrow F & \downarrow F \\
\text{Id} & \gamma & \text{Id}
\end{array}
\]

be commutative. A map of \( F \)-algebras is a map of objects in \( M \) that is compatible with the structure maps.

1.6. Generalized Props

Here we define generalized props associated to a pasting scheme as algebras over a monad. Recall that a category is said to be \textit{(co)complete} if it has all small \textit{(co)limits}.

\begin{convention}
Throughout the rest of this book, we work over a cocomplete symmetric monoidal category

\((M, \otimes, \mathbb{1})\)

whose monoidal product \( \otimes \) commutes with colimits on both sides. We usually just write an unordered tensor product as \( \otimes \). If we want to emphasize an unordered tensor product, then we will use the symbol \( \odot \).
\end{convention}

The commutation of \( \otimes \) with colimits is automatic if \( M \) is closed \cite{Mac98} VII.7. An empty tensor product is by definition the monoidal unit \( \mathbb{1} \). If \( D \) is a small category, then \( M^D \) denotes the diagram category of functors \( D \to M \) with natural transformations as morphisms. The \textit{discrete category} associated to a category \( C \), denoted by \( \text{dis}(C) \), has the same objects as \( C \) and only identity morphisms.

In a graph \( G \), the set \( Vt(G) \) of vertices is not ordered. If we want to define a tensor product indexed by the vertices in a graph, then we need to use a set-indexed or unordered tensor product as follows.
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**Definition 1.6.2.** Suppose \( \mathcal{G} = (S, G) \) is a pasting scheme, and \( P \in M^{\text{dis}}(S) \). For each \( G \in \mathcal{G} \), define the object

\[
P[G] = \bigotimes_{v \in G} P(v) = \left( \bigotimes_{\varphi \in \text{Ord}(Vt(G))} \bigoplus_{j=1}^k P(\varphi(j)) \right)_{\Sigma_k} \in \mathcal{M}
\]

in which:

- \( P(v) = P(\text{Prof}(v)) \) for each vertex \( v \) in \( G \), and \( k = |Vt(G)| \).
- \( \text{Ord}(Vt(G)) \) is the set of bijections \( \{1, \ldots, k\} \rightarrow Vt(G) \).
- Each \( \tau \in \Sigma_k \) acts on the coproduct by sending the \( \varphi \)-summand to the \( \varphi \tau \)-summand.
- \( (\cdot \cdot \cdot)_{\Sigma_k} \) is the \( \Sigma_k \)-coinvariant.

We call \( P[G] \) a \( P \)-decorated graph or a \( P \)-decoration of \( G \), and call \( \bigotimes_{v \in G} \) the unordered tensor product indexed by the set of vertices in \( G \).

**Example 1.6.3.** Suppose \( P \in M^{\text{dis}}(S) \).

1. For a disjoint union, we have

\[
P[G_1 \sqcup \cdots \sqcup G_n] = \bigotimes_{j=1}^n P[G_j].
\]

2. For a graph substitution \( G(H_v) \), we have

\[
P[G(H_v)] = \bigotimes_{v \in G} P[H_v].
\]

3. Since there are no vertices in the empty graph \( \emptyset \), the exceptional edge \( \uparrow_e \), and the exceptional loop \( \bigcirc_c \), we have

\[
\]

4. For the permuted corolla \( \sigma C_{(\leq d)} \tau \) in Example 1.2.19, the vertex profile is \( (\leq d) \), so

\[
P[\sigma C_{(\leq d)} \tau] = P(d).
\]

Note that this includes the case \( (\tau; \sigma) = (\text{Id}; \text{Id}) \).

5. Likewise, for the contracted corolla in Example 1.2.20, we have

\[
P[\zeta_k C_{(\leq d)}] = P(d).
\]

6. For the one-colored graph \( G \) in Example 1.2.8, we have

\[
P[G] = P(u) \odot P(v) \odot P(w) = P(4; 2) \odot P(1; 4) \odot P(1; 1).
\]

7. For the one-colored graph \( G \) in 1.3.3, we have

\[
P[G] = P(u) \odot P(v) \odot P(w) = P(2; 2) \odot P(2; 1) \odot P(1; 2).
\]

**Definition 1.6.4.** For a pasting scheme \( \mathcal{G} = (S, G) \) and \( (d_\xi, c_{(\leq d)}) \in S \), denote by \( G(d_\xi) \) the full sub-groupoid containing \( G \in \mathcal{G} \) with \( \text{Prof}(G) = (d_\xi, c_{(\leq d)}) \).

We now define the monad whose algebras are \( \mathcal{G} \)-props.

**Definition 1.6.5.** Suppose \( \mathcal{G} = (S, G) \) is a \( C \)-colored pasting scheme. Define a monad \( (F, \mu, \nu) \) on \( M^{\text{dis}}(S) \) as follows.
The functor: Define the functor

$$F : M_{\text{dis}}(S) \longrightarrow M_{\text{dis}}(S)$$

by

$$FP(\hat{\ell}) = \bigsqcup_{[G] \in G(\hat{\ell})} P[G]$$

for $P \in M_{\text{dis}}(S)$ and $\hat{\ell} \in S$, where the coproduct is taken over the isomorphism classes in $G(\hat{\ell})$.

The multiplication: For each $G \in G$, we have

$$FP[G] = \bigotimes_{v \in G} FP(v)$$

$$= \bigotimes_{v \in G} \bigsqcup_{[H_v] \in G(v)} P[H_v]$$

$$\cong \bigsqcup_{\{H_v\} \in F[\hat{\ell}]} \bigotimes_{v \in G} P[H_v].$$

The monadic multiplication $\mu_P$ is defined by the commutative diagrams

$$FP(\hat{\ell}) = \bigsqcup_{[G] \in G(\hat{\ell})} P[G]$$

$$\xrightarrow{\mu_P} \bigotimes_{v \in G} P[H_v]$$. 

$$
\begin{array}{c}
\begin{array}{c}
\bigsqcup_{[G] \in G(\hat{\ell})} \\
\bigotimes_{[v \in G [H_v] \in G(v)]} P[H_v]
\end{array}
\end{array}
\xrightarrow{\text{inclusion}}
\begin{array}{c}
\bigotimes_{v \in G} P[H_v] \\
\end{array}
\xrightarrow{\text{inclusion}}
\begin{array}{c}
\bigotimes_{v \in G} P[H_v] \\
\end{array}
\xrightarrow{\mu_P}
\begin{array}{c}
\bigotimes_{v \in G} P[H_v]
\end{array}
$$

for $\hat{\ell} \in S$, $G \in G(\hat{\ell})$, and $\{H_v\} \in \prod_{v \in G} G(v)$.

The unit: The monadic unit is defined by the corolla inclusion

$$P(\hat{\ell}) = P[C(\hat{\ell})]$$

in which $C(\hat{\ell})$ is the $(\hat{\ell}, \hat{\ell})$-corolla.

The proof that $(F, \mu, \nu)$ is a monad is in [YJ15] Theorem 10.38. Associativity and unity of the monad are consequences of those of graph substitution.

**Definition 1.6.6.** For a $\mathcal{C}$-colored pasting scheme $\mathcal{G}$, the category of algebras over the monad $(F, \mu, \nu)$ on $M_{\text{dis}}(S)$ is denoted by $\text{Prop}^\mathcal{G}(M)$. Its objects are called $\mathcal{G}$-props in $M$.

**Example 1.6.7.** For a non-empty set $\mathcal{C}$:

1. $\text{Gr}_{\mathcal{Q}}$-props are $\mathcal{C}$-colored wheeled props in $M$.
2. $\text{Gr}_{\mathcal{C}}$-props are $\mathcal{C}$-colored wheeled properads in $M$.
3. $\text{Tree}_{\mathcal{Q}}$-props are $\mathcal{C}$-colored wheeled operads in $M$.
4. $\text{Gr}^1$-props are $\mathcal{C}$-colored props in $M$. 

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(5) $Gr^1$-props are $\mathcal{C}$-colored properads in $M$.
(6) $Gr^\omega$-props are $\mathcal{C}$-colored dioperads in $M$.
(7) UTree-props are $\mathcal{C}$-colored operads in $M$.
(8) ULin-props are small categories enriched in $M$ with object set $\mathcal{C}$. In particular, if $\mathcal{C}$ is the one-point set, then ULin-props are monoids in $M$.
(9) Cor$_S$-props are $S$-diagrams in $M$.

Convention 1.6.8. To simplify notation, we will take isomorphism classes tacitly and simply write $G$ for $[G]$.

Unraveling the definition of the monad $(F, \mu, \nu)$, we have the following more explicit description of a $G$-prop.

Proposition 1.6.9. For a $\mathcal{C}$-colored pasting scheme $G$, a $G$-prop is exactly a pair $(P, \gamma^P)$ consisting of:
- an object $P \in M^{\operatorname{dis}(S)}$;
- a structure map $P[G] \xrightarrow{\gamma^P_G} P(\frac{d}{c})$ for each $(\frac{d}{c}) \in S$ and each $G \in G(\frac{d}{c})$.

This data is subject to the following two conditions.

Unity: For each $(\frac{d}{c}) \in S$, the structure map $\gamma^P_{G(\frac{d}{c})}$ is the identity map of $P(\frac{d}{c})$.

Associativity: The diagram

\[
\begin{array}{ccc}
\otimes_{v \in G} P[H_v] & \xrightarrow{\otimes_{v \in G} \gamma^P_G[H_v]} & \otimes_{v \in G} P(v) = P[G] \\
\downarrow \circlearrowleft & & \downarrow \circlearrowleft \\
S & \xrightarrow{\gamma^P_G[P(G(H_v))]} & P(\frac{d}{c})
\end{array}
\]

is commutative for all $(\frac{d}{c}) \in S$, $G \in G(\frac{d}{c})$, and $(H_v) \in \prod_{v \in G} G(v)$.

Furthermore, a map of $G$-props $f : (P, \gamma^P) \longrightarrow (Q, \gamma^Q)$ is exactly a map $f : P \longrightarrow Q \in M^{\operatorname{dis}(S)}$ such that the diagram

\[
\begin{array}{ccc}
P[G] & \xrightarrow{\otimes_{v \in G} f} & Q[G] \\
\gamma^P_G & & \gamma^Q_G \\
P(\frac{d}{c}) & \xrightarrow{f} & Q(\frac{d}{c})
\end{array}
\]

is commutative for all $(\frac{d}{c}) \in S$ and $G \in G(\frac{d}{c})$.

Example 1.6.11. Suppose $P$ is a $G$-prop for a pasting scheme $G = (S, G)$.

1. The structure maps $P[C(\frac{d}{c})^\tau] = P(\frac{d}{c}) \xrightarrow{\gamma^P_{C(\frac{d}{c})^\tau}} P(\frac{d}{c})$
for permuted corollas with \( \frac{c}{d} \in S \) yield the equivariant structure on the \( G \)-prop \( P \).

(2) If the exceptional edge \( \uparrow c \) belongs to \( G \), then the structure map

\[
P[\uparrow c] = 1 \xrightarrow{\gamma_P \uparrow c} P(\) \]

is the \( c \)-colored unit of \( P \).

(3) If the contracted corolla \( \xi_k \) \( C_{\{d;\}} \) belongs to \( G \), then the structure map

\[
P[\xi_k \downarrow C_{\{c;\}}] = P(\xi_k \downarrow C_{\{d;\}}) \xrightarrow{\gamma_P \xi_k \downarrow C_{\{c;\}}} P(\) \]

is usually called a contraction of \( P \).

(4) For the one-colored graph \( G \) in Example 1.2.8 we have the structure map

\[
P[G] = P(4;2) \odot P(1;4) \odot P(1;1) \xrightarrow{\gamma_G} P(1;2) \] .

(5) For the one-colored graph \( G \) in (1.3.3), we have the structure map

\[
P[G] = P(2;2) \odot P(2;1) \odot P(1;2) \xrightarrow{\gamma_G} P(1;1) .
\]

Remark 1.6.12. For each pasting scheme \( G \) in use, there is also a coherence theorem that describes \( G \)-props in terms of just a few generating structure maps \( \gamma_P \) (as opposed to \( \gamma_G \) for all \( \frac{c}{d} \in S \) and for all \( G \in G_{\{c;\}} \)) and a few generating axioms (as opposed to, say, the associativity axiom for all possible graph substitution in \( G \)). We refer the reader to [YJ15] Chapter 11 for details. In what follows, since our Boardman-Vogt construction applies to a general pasting scheme, we will only need the description of a \( G \)-prop in Prop. 1.6.9.

1.7. Changing the Pasting Schemes

The following material on change of pasting schemes is from [YJ15] Chapter 12.

Theorem 1.7.1. For each inclusion \( G \subseteq G' \) of pasting schemes, there is a free-forgetful adjunction

\[
\text{Prop}^G(M) \xrightarrow{F^G, G'} \text{Prop}^{G'}(M)
\]

in which the right adjoint \( U \) is the forgetful functor.

The left adjoint can be explicitly described using the following indexing category.

Definition 1.7.3. Suppose given a pair

\( G = (S, G) \subseteq (S', G') = G' \)

of pasting schemes and a pair \( \frac{c}{d} \in S' \). The extension category \( D_{\{c;\}} \) is the small category defined as follows.

- Its objects are strict isomorphism classes of graphs in \( G'_{\{c;\}} \) in which each vertex profile lies in \( S \).
A map has the form

\[(H_v)_{v \in K} : K(H_v) \longrightarrow K\]

with each \(H_v \in G(v)\).

Composition is defined by graph substitution in \(G\):

\[K[H_v(I_v^u)] = [K(H_v)](I_v^u) \xrightarrow{(I_v^u)} K(H_v) \xrightarrow{(H_v)} K\]

Identities are families of corollas

\[(C_v)_{v \in K} : K = K(C_v) \longrightarrow K\]

with each \(C_v \in G(v)\) the \(v\)-corolla.

The following observation is from [YI15] (Lemma 12.6 and Lemma 12.8).

**Lemma 1.7.4.** The left adjoint

\[F^G : \text{Prop}^G(M) \longrightarrow \text{Prop}^{G'}(M)\]

is given by

\[F^G X(\delta) = \colim_{G \in \delta} X[G]\]

for any \(G\)-prop \(X\) and pair \((\delta) \in S'\).

We will use this Lemma several times later in this book.
CHAPTER 2

Classical Boardman-Vogt Construction for Operads

As motivation and history of our Boardman-Vogt construction of generalized props, in this chapter we recall the \( W \)-construction of topological colored operads defined in \cite{BV72} Chapter III and \cite{Vog03}. Throughout this chapter, the underlying monoidal category is that of compactly generated topological spaces. The book \cite{Yau16} provides a leisurely introduction to colored operads.

2.1. Utility of the Boardman-Vogt Construction

Suppose \( \mathcal{O} \) is a \( \mathcal{C} \)-colored operad. The objective is to construct a \( \mathcal{C} \)-colored operad \( W\mathcal{O} \) together with a natural augmentation \( \eta : W\mathcal{O} \longrightarrow \mathcal{O} \) of operads such that:

1. \( \eta \) is entrywise a weak homotopy equivalence.
2. If \( \mathcal{O} \) is sufficiently nice, then \( W\mathcal{O} \) is a cofibrant topological operad.

The augmentation \( \eta \) induces an adjunction \( \eta ! \dashv \eta * \) between the categories of \( W\mathcal{O} \)-algebras and \( \mathcal{O} \)-algebras. If \( \mathcal{O} \) and \( W\mathcal{O} \) are nice enough, then the fact that \( \eta \) is a weak homotopy equivalence implies the adjunction \( \eta ! \dashv \eta * \) is a Quillen equivalence, so induces an equivalence between homotopy categories. Conditions under which this happens can be found in, e.g., \cite{BM03} Theorem 4.4 and \cite{WY}. The second condition—i.e., the cofibrancy of \( W\mathcal{O} \)—implies that \( W\mathcal{O} \)-algebra is a homotopy invariant structure in the following sense.

Suppose \( f : X \longrightarrow Y \) is a weak equivalence and \( X \) (resp., \( Y \)) is a \( W\mathcal{O} \)-algebra. Then there is an essentially unique \( W\mathcal{O} \)-algebra structure on \( Y \) (resp., \( X \)) such that \( f \) becomes a weak equivalence of \( W\mathcal{O} \)-algebras.

In other words, without changing the weak equivalence type of the operad \( \mathcal{O} \), we have an operad \( W\mathcal{O} \) whose algebras are homotopy invariant. Homotopy invariance of algebras over cofibrant operads is explained in \cite{BM03} Theorem 3.5. For algebras over cofibrant (generalized) props, homotopy invariance is explained in \cite{JY09} Theorem 1.2 and \cite{YJ15} Cor. 13.23.

2.2. Free Operads

The operad \( W\mathcal{O} \) for a \( \mathcal{C} \)-colored operad \( \mathcal{O} \) is constructed by adapting the free operad construction. See, e.g., \cite{Yau16} Part 4 for details about free operads.
Suppose \( X(\underline{c}) \) is a space for each \( d, c_1, \ldots, c_n \in \mathfrak{C} \), where \( \underline{c} = (c_1, \ldots, c_n) \). Then the free \( \mathfrak{C} \)-colored operad of \( X \) has \( \underline{c} \)-entry

\[
(FX)(\underline{c}) = \prod_T X[T]
\]

where the coproduct is taken over the set of isomorphism classes of unital trees with profile \( \underline{c} \). The object \( X[T] \) is the product

\[
X[T] = \prod_{v \in T} X(v)
\]

and is called the \( X \)-decoration of \( T \). Here \( v \in T \) means \( v \) runs through the set of vertices in \( T \), and \( X(v) \) is the entry of \( X \) whose profile is that of \( v \).

**Example 2.2.1.** Suppose \( T \) is the unital tree

\[
\text{(2.2.2)}
\]

in which, for simplicity, the incoming edges at each vertex and the legs are all ordered from left to right. The color of each edge is indicated near the edge. Then \( T \) has profile \( (a, b, d, e, h) \). The \( X \)-decoration of \( T \) is the product

\[
X[T] = X(u) \times X(v) \times X(w) \times X(x)
\]

\[
= X(\underline{a}) \times X(\underline{b}) \times X(\underline{d}) \times X(b).
\]

The operad structure on \( FX \) is induced by grafting of unital trees, i.e., connecting the output of a unital tree with an input leg of another unital tree with the same color. The equivariant structure on \( FX \) corresponds to permuting the input leg ordering. For each color \( c \in \mathfrak{C} \), the \( c \)-colored unit in \( FX \) corresponds to the exceptional edge \( \uparrow \) with color \( c \) and trivial \( X \)-decoration.

**Example 2.2.3.** Suppose \( T \) is the unital tree in \((2.2.2)\) and \( T' \) is the corolla

\[
\text{(2.2.4)}
\]

with profile \( \underline{b} \). Then the operadic comp-2 operation

\[
(FX)(\underline{a, b, d, e, h}) \times (FX)(\underline{b}) \xrightarrow{e_2} (FX)(\underline{a, b, d, e, c, b})
\]

sends \((X[T], X[T'])\) to \( X[T \circ_2 T'] \). Here \( T \circ_2 T' \) is the unital tree
obtained by grafting the output of $T'$ with the second input leg of $T$.

2.3. Topological Boardman-Vogt Construction

The $W$-construction of a $\mathcal{C}$-colored operad $O$ starts with the free operad on the underlying object of $O$, but each internal edge now also has an assigned length from the interval $J = [0, 1]$. The output leg and the input legs are all given length 1. The interval $J$ is given the monoid structure with multiplication

$$s \ast t = 1 - (1 - s)(1 - t).$$

So 0 is the unit for $\ast$, and 1 is absorbing in the sense that $s \ast 1 = 1 = 1 \ast s$. Alternatively, one may also define $s \ast t = \max\{s, t\}$.

Example 2.3.1. The unital tree $T$ in (2.2.2) yields the space

$$(J \times O)[T] = J^3 \times O(u) \times O(v) \times O(w) \times O(x)$$

in which $[T]$ denotes the set of internal edges in $T$. Likewise, for the corolla $T'$ in (2.2.4), we have

$$(J \times O)[T'] = O_+.$$  

For a general unital tree $T$, we will refer to $(J \times O)[T]$ as a space of decorated trees and a point in it as a decorated tree. We now impose three kinds of identification among these decorated trees, corresponding to the equivariance, unit, and operadic composition of $O$.

Equivariant Identification: Suppose $p \in O$ such that $p \sigma \in O_\sigma$ decorates a vertex $v$ in a decorated tree. This decorated tree is identified with the one obtained by replacing $v$ with $v \sigma^{-1}$, in which the ordering $\rho$ of the set of incoming edges at $v$ is replaced by $\rho \sigma^{-1}$, with decoration $p$.

\[ (2.3.2) \]

Unit Identification: Suppose the $c$-colored unit $1_c \in O(\cdot)$ decorates a vertex $v$ in a decorated tree. This decorated tree is identified with the one obtained by removing the vertex $v$ and using the monoid structure on the interval
for the length of the resulting edge.

(2.3.3)

**Compositional Identification:** Suppose \( p \in \mathcal{O}(\underline{d}) \) with \(|d| \geq 1\) and \( q \in \mathcal{O}(\underline{c}) \) decorate two vertices of a decorated tree that are connected by an edge \( e \) with length 0. This decorated tree is identified with the one obtained by collapsing the edge \( e \) and decorating the resulting vertex by the operadic composition \( p \circ_i q \).

Note that each of the three kinds of identification preserves the profiles of the unital trees involved.

The \((\underline{d})\)-entry of \( \mathcal{W}O \) is defined as the quotient space

(2.3.5)

\[
(WO)_{\underline{d}} = \left( \coprod_T (J \times \mathcal{O}(\underline{d})) / \sim \right)
\]

in which the coproduct is taken over the set of isomorphism classes of unital trees with profile \((\underline{d})\). The quotient is generated by the three kinds of identification above.

**Example 2.3.6.** For the unital tree \( T \) in (2.2.2), suppose \( p_u \in \mathcal{O}(u) \), \( p_v \in \mathcal{O}(v) \), \( p_w \in \mathcal{O}(w) \), and \( p_x \in \mathcal{O}(x) \). The two decorated trees \( A' \) and \( A \),

represent the same point in \( \mathcal{W}O_{\underline{a,b,d,c,b}} \). Here the lengths of the internal edges are indicated. Since input/output legs are always given length 1, they are not drawn in the picture. On the left side, the middle internal edge has length 0. So by the compositional identification, the left decorated tree is identified with the right decorated tree, where the top vertex is decorated by the operadic composition \( p_u \circ_2 p_w \).

The operad structure on \( \mathcal{W}O \) is defined just like the free operad above using grafting of unital trees, with new internal edges arising from an operadic composition all given length 1. The colored units are represented by the points
(J \times O)[\uparrow] = \ast. The equivariant structure comes from permutations of the input legs of trees.

**Example 2.3.7.** Consider the right decorated tree $A$ in Example 2.3.6 and the decorated corolla

$$B = \begin{array}{c}
\text{decorated corolla} \\
b \\
d \\
c \\
\ast
\end{array}$$

given by $T'$ in [2.2.4] with $p_y \in O(y)$ decorating its unique vertex. Then the operadic comp-$2$ operation

$$(W O)(\underbrace{\cdots}_{a,b,d,c,b}) \times (W O)(\underbrace{\cdots}_{b,d,c}) \xrightarrow{\circ_2} (W O)(\underbrace{\cdots}_{a,d,c,d,c,b})$$

sends $(A,B)$ to the decorated tree:

$$A \circ_2 B = \begin{array}{c}
\text{decorated tree} \\
0.5 \\
0.7 \\
0.1 \\
0.9 \\
d \\
c \\
\ast
\end{array}$$

(2.3.9)

Here the new internal edge created by grafting is assigned length 1.

### 2.4. Augmentation

The augmentation $\eta : W O \rightarrow O$ is defined by forgetting the lengths of the internal edges and composing operadically in $O$. More precisely, for each unital tree $T$ with profile \((\cdot)\), there is an operadic structure map

$$O[T] = \prod_{v \in T} O(v) \xrightarrow{\gamma_T} O(\cdot).$$

The maps

$$(J \times O)[T] \xrightarrow{\eta} (\ast \times O)[T] = O[T] \xrightarrow{\gamma_T} O(\cdot)$$

induced by $J \rightarrow \ast$ yield the desired augmentation $\eta$ in the \((\cdot)\)-entry. Since the map $J \rightarrow \ast$ is a weak homotopy equivalence, so is each entry of the augmentation $\eta$.

**Example 2.4.1.** Continuing Example 2.3.7, the augmentation

$$(W O)(\underbrace{\cdots}_{a,d,c,d,c,b}) \xrightarrow{\eta} O(\underbrace{\cdots}_{a,d,c,d,c,b})$$

sends $A \circ_2 B$ in (2.3.9) to the iterated operadic composition

$$\left[((p_u \circ_2 p_w) \circ_1 p_v) \circ_2 p_y \circ_5 p_x\right]$$

in $O$. 
2.5. Filtration

Each \( \binom{n}{d} \)-entry of the operad \( WO \) has a natural filtration by sub-spaces
\[
(WO)_0(\binom{n}{d}) \subseteq (WO)_1(\binom{n}{d}) \subseteq (WO)_2(\binom{n}{d}) \subseteq \cdots \subseteq (WO)(\binom{n}{d}).
\]
For \( n \geq 0 \) the \( n \)th filtration is defined as the quotient space
\[
(WO)_n(\binom{n}{d}) = \left( \bigsqcup_{|T| \leq n} (J \times O)[T] \right) / \sim
\]
in which the coproduct is indexed by the set of isomorphism classes of unital trees with profile \( \binom{n}{d} \) and at most \( n \) internal edges. The identification are the same as before, suitably restricted to each filtration stratum. When \( O \) is sufficiently nice, the cofibrancy of the operad \( WO \) is proved by analyzing this filtration.

**Example 2.5.1.** Continuing Examples \[2.3.6\] and \[2.3.7\] we have that
\[
A \in (WO)_2(\binom{2}{a,d,c,d,c,b}) \quad \text{and} \quad A \circ_2 B \in (WO)_3(\binom{3}{a,d,c,d,c,b}).
\]
The left decorated tree \( A' \) in Example \[2.3.6\] has three internal edges and is in \( (WO)_3(\binom{3}{a,d,c,d,c,b}) \). It is identified with \( A \), which belongs to the previous stratum \( (WO)_2 \), by the compositional identification. This example illustrates that a decorated tree in a stratum \( (WO)_{n+1} \) can be identified with a decorated tree in the previous stratum \( (WO)_n \) if it has either

1. an internal edge with length 0 (via the compositional identification) or
2. a vertex decorated by a colored unit of \( O \) (via the unit identification).

This phenomenon is the key to relating two consecutive strata in the filtration of \( WO \).
CHAPTER 3

Boardman-Vogt Construction of Generalized Props

In this chapter, we define the Boardman-Vogt construction, also called the $W$-construction, for each generalized prop associated to a pasting scheme and for each choice of a commutative segment in the ambient category. We observe that the $W$-construction has a natural generalized prop structure. Further categorical and homotopical properties will be studied in later chapters.

Suppose $(M, \otimes, 1)$ is a cocomplete symmetric monoidal category in which the monoidal product commutes with colimits in both variables, which is automatically true if $M$ is also closed. Later we will assume that $M$ is a monoidal model category, but a model structure is not needed for the $W$-construction itself. Fix a $C$-colored pasting scheme $G = (S, G)$. With $S$ understood, we will not mention it again, and write $G$ as $\mathcal{G}$. When we consider a pair of $C$-profiles, we mean it is an element in $S$.

In Section 3.1 we define the substitution category that parametrizes the co-end in the definition of the $W$-construction. In Section 3.2 we recall the definition of a commutative segment, which yields the contravariant variable of the $W$-construction. Some categorical properties of coends are discussed in Section 3.3. The $W$-construction of a $\mathcal{G}$-prop is defined in Section 3.4. Its $\mathcal{G}$-prop structure is discussed in Section 3.5.

3.1. Substitution Category

We will use only strict isomorphism classes of graphs, which we will simply refer to as graphs. In particular, graph substitution is strictly associative and unital. If $G \in \mathcal{G}(\mathcal{T})$ and if $v$ is a vertex in $G$, then we write $(v)$ for the pair $\langle \text{out}(v) \rangle$ of $C$-profiles and $\mathcal{G}(v)$ for $\mathcal{G}(\text{out}(v))$. We write $|G|$ for the set of ordinary internal edges in $G$. The notation $v \in G$ means that $v$ runs through the set of vertices in $G$.

Motivation 3.1.1. In order to define the Boardman-Vogt construction for a generalized prop, we need a more structured way to present the kinds of identification in Section 2.3 that can be applied uniformly across all pasting schemes. The equivariant, unit, and compositional identifications in Section 2.3 can all be understood via graph substitution of unital trees. Indeed, the equivariant identification involves graph substitution of permuted corollas. The unit identification involves graph substitution of exceptional edges. The compositional identification involves graph substitution of unital trees with at least one internal edge. All unital trees are involved in these identifications because a unital tree either has at least one internal edge or has no internal edges. In the latter case, it is either a permuted corolla (i.e., has one vertex) or an exceptional edge (i.e., has no vertices).

Graph substitution is a common feature of all pasting schemes. This suggests that, for the Boardman-Vogt construction of generalized props, we should use graph
substitution as a device to parametrize the desired identification. The substitution category in the next definition is the precise graph substitution device that will be used to define the Boardman-Vogt construction of generalized props.

**Definition 3.1.2.** Suppose \((\mathcal{C}, \mathcal{D})\) is a pair of \(\mathcal{C}\)-profiles, and \(\mathcal{P}\) is a \(\mathcal{G}\)-prop in \(\mathcal{M}\).

1. Define the **substitution category** \(\mathcal{G}(\mathcal{C}, \mathcal{D})\) as the extension category \(\mathcal{D}(\mathcal{C}, \mathcal{D})\) in Def. 1.7.3 for the identity inclusion \(\mathcal{G} \leq \mathcal{G}\). In other words, its objects are graphs \(G \in \mathcal{G}(\mathcal{C}, \mathcal{D})\). A map \((H_v) : G(H_v) \to G\) in \(\mathcal{G}(\mathcal{C}, \mathcal{D})\) is given by a family of graphs \(H_v \in \mathcal{G}(v)\) as \(v\) runs through the vertices in \(G\). The identity map of an object \(G\) is \((C_v) : G = G(C_v) \to G\), where \(C_v\) is the corolla whose profile is equal to that of \(v \in G\). Composition of maps is given by graph substitution.

2. Define a functor \(\mathcal{P} : \mathcal{G}(\mathcal{C}, \mathcal{D}) \to \mathcal{M}\) by setting \(\mathcal{P}[G] = \bigotimes_{v \in G} \mathcal{P}(v)\) for each object \(G \in \mathcal{G}(\mathcal{C}, \mathcal{D})\). For a map \((H_v) : G(H_v) \to G \in \mathcal{G}(\mathcal{C}, \mathcal{D})\), define the map

\[
\mathcal{P}[G(H_v)] = \bigotimes_{v \in G} \mathcal{P}[H_v] \quad \text{by} \quad \bigotimes_{v \in G} \mathcal{P}(v) = \mathcal{P}[G]
\]

in which \(\gamma^P_{H_v} : \mathcal{P}[H_v] \to \mathcal{P}(v)\) is the \(\mathcal{G}\)-prop structure map of \(\mathcal{P}\) corresponding to \(H_v \in \mathcal{G}(v)\). This assignment preserves the identity maps because \(\gamma^P_C\) is the identity map whenever \(C\) is a corolla. It preserves composition of maps by the associativity of \(\gamma^P_{H_v}\).

**Example 3.1.3.** Let us consider one of the simplest substitution categories. Suppose \(\mathcal{C}\) is a singleton \(*\) and \(\mathcal{ULin}\) is the one-colored pasting scheme of unital linear graphs. Since there is only one pair of \(\mathcal{C}\)-profiles where the input and the output profiles both have length 1, we abbreviate the substitution category to \(\mathcal{ULin}\). Then \(\mathcal{ULin}\) is isomorphic to the opposite of the wide subcategory of the ordinal number category of functors that preserve both the initial and the terminal objects.

In more details, a one-colored unital linear graph with \(n \geq 0\) vertices \(L_n\) is exactly a linearly ordered set

\([n] = \{0 < 1 < \cdots < n\}\).

So \([0]\) corresponds to the exceptional edge \(L_0 = \uparrow\). Here is a depiction of the linear graph \(L_4\):

\[
L_4 = [4] = \begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
\uparrow & 1 & 2 & 2 & 4
\end{array}
\]
In each \([n]\) we refer to 0 and \(n\) as extreme objects. Recall the ordinal number category \(\Delta\) with objects the categories \([n]\) for \(n \geq 0\) and functors (i.e., weakly order-preserving maps) as morphisms. Denote by \(D\) the wide subcategory of \(\Delta\) of extreme-preserving functors. In other words:

- \(D\) contains all the objects in \(\Delta\).
- \(D([m],[n])\) consists of the functors \(f : [m] \rightarrow [n] \in \Delta\) such that
  \[
  f(0) = 0 \quad \text{and} \quad f(m) = n.
  \]

Then the substitution category \(ULin\) is the opposite category of \(D\).

For instance, consider the extreme-preserving functor \(f : [3] \rightarrow [5] \in D\) given by

\[
\begin{align*}
f(0) &= 0, \\
f(1) &= f(2) = 2, \quad \text{and} \quad f(3) = 5.
\end{align*}
\]

In the substitution category \(ULin\), \(f\) corresponds to the map

\[
(3.1.4) \quad (H_v) : L_5 = L_3(H_v) \rightarrow L_3
\]

in which

\[
H_1 = L_2, \quad H_2 = \uparrow, \quad \text{and} \quad H_3 = L_3.
\]

This map is a way to construct the linear graph \(L_5\) out of \(L_3\) via graph substitution, as depicted in the following picture of \(L_5\).

Here the linear graphs inside the gray boxes are, from left to right, \(H_1 = L_2\), \(H_2 = \uparrow\), and \(H_3 = L_3\), respectively.

A \(ULin\)-prop \(P\) is a category enriched in \(M\) with object set \(\mathcal{C} = \ast\), i.e., a monoid in \(M\). For each \(n \geq 2\), the structure map

\[
P[n] = P \otimes^n \gamma_p^{[n]} \rightarrow P
\]

is the \((n - 1)\)-fold iterate of the multiplication in \(P\), with

\[
\gamma_p^{[0]} : 1 \rightarrow P
\]

the unit and \(\gamma_p^{[1]}\) the identity map of \(P\).

**Example 3.1.5.** Consider the pasting scheme \(Gr^c\) of \(\mathcal{C}\)-colored connected wheel-free graphs, so \(Gr^c\)-props are \(\mathcal{C}\)-colored properads. Suppose \(a, b, \ldots, f \in \mathcal{C}\). Consider the connected wheel-free graphs \(G, H_u, H_v = \uparrow_c, H_w,\) and \(K = G(H_u, H_v, H_w)\):
For simplicity, in $G$, $H_u$, and $H_w$, all the listings at the vertices and for the whole graphs are from left to right as displayed. The gray boxes and the gray arrows indicate graph substitution.

In the substitution category $\text{Gr}^c_\downarrow\uparrow$, there is a map

$$K = G(H_u, H_v, H_w) \xrightarrow{(H_u, H_v, H_w)} G.$$ (3.1.6)

If $P$ is a properad in $M$, then the corresponding map is the unordered tensor product

$$P[K] = P(w_1) \otimes P(w_2) \otimes P(u_1) \otimes P(u_2) \otimes \mathbb{1}$$

$$\gamma_{H_u} \otimes \gamma_{H_v} \otimes \gamma_{H_w}$$

$$P[G] = P(w) \otimes P(u) \otimes P(v)$$

in $M$.

**Example 3.1.8.** Consider the pasting scheme $\text{Gr}^c_\downarrow$ of $C$-colored connected wheeled graphs, so $\text{Gr}^c_\downarrow$-props are $C$-colored wheeled properads. Suppose $a, b, c, d \in C$. Consider the graphs $G$, $H_u$, $H_v = \uparrow_c$, and their graph substitution $K = G(H_u, H_v)$:

For $H_u$, the output leg with color $a$ is ordered first, and the vertex listings at $x$ and $y$ are arbitrary. At the vertex $u$ in $G$, the leg with color $a$ is ordered first. The gray boxes and the gray arrows indicate graph substitution. There is a map

$$K = G(H_u, H_v) \xrightarrow{(H_u, H_v)} G$$ (3.1.9)
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in the substitution category $\mathcal{G}^{\mathcal{C}}_e(\mathcal{C})$. For a wheeled properad $P$ in $\mathcal{M}$, the map $P[K] \longrightarrow P[G]$ is the unordered tensor product

\[(3.1.10) \quad P[K] \cong P(x) \otimes P(y) \otimes 1 \xrightarrow{\gamma} P(u) \otimes P(v) = P[G].\]

**Example 3.1.11 (Entrywise Monoidal Unit).** Suppose $G$ is a $\mathcal{C}$-colored pasting scheme. There is a $G$-prop $\text{Com}^G$ in which every entry is the monoidal unit $1$ in $\mathcal{M}$. Every structure map

$$\text{Com}^G[G] \cong 1 \longrightarrow 1 = \text{Com}^G(\mathcal{C})$$

for $G \in \mathcal{G}(\mathcal{C})$ is the identity map. When $G$ is the one-colored pasting scheme of unital trees (so $G$-props are operads in $\mathcal{M}$), the $G$-prop $\text{Com}^G$ is called the *commutative operad*, whose algebras are exactly the unital commutative monoids in $\mathcal{M}$.

**Example 3.1.12 (Terminal Generalized Prop).** Suppose $G$ is a $\mathcal{C}$-colored pasting scheme. The terminal $G$-prop $T$ is entrywise the terminal object $\ast$ in $\mathcal{M}$. So every structure map

$$\text{T}[G] = \bigotimes_{v \in G} \text{T}(v) \xrightarrow{\gamma} \ast = \text{T}(\mathcal{C})$$

for $G \in \mathcal{G}(\mathcal{C})$ is the unique map to the terminal object.

**Example 3.1.13 (Initial Generalized Prop).** Suppose $G$ is a $\mathcal{C}$-colored pasting scheme. The initial $G$-prop $I$ has entries

$$I(\mathcal{C}) = \bigcup_{G \in \mathcal{G}(\mathcal{C})} 1$$

in which $\mathcal{G}(\mathcal{C})$ is the set of graphs in $\mathcal{G}(\mathcal{C})$ with no vertices. For example:

1. Suppose $G \leq \text{Gr}_{\mathcal{C}}$ is unital, so all the graphs in $G$ are connected and wheel-free. This includes the $\mathcal{C}$-colored pasting schemes of unital linear graphs (for small $\mathcal{M}$-enriched categories with object set $\mathcal{C}$), of trees (for $\mathcal{C}$-colored operads in $\mathcal{M}$), of simply-connected graphs (for $\mathcal{C}$-colored dioperads in $\mathcal{M}$), and of all connected wheel-free graphs (for $\mathcal{C}$-colored properads in $\mathcal{M}$). Then the initial $G$-prop has entries

$$I(\mathcal{C}) = \begin{cases} \emptyset & \text{if } (\mathcal{C}) \neq (\mathcal{C}) \text{ for some } c \in \mathcal{C} ; \\ 1 & \text{if } (\mathcal{C}) = (\mathcal{C}) \text{ for some } c \in \mathcal{C} . \end{cases}$$

Its only non-trivial structure maps are the isomorphisms

\[(3.1.14) \quad I[L_n] = 1 \otimes 1 \xrightarrow{\gamma} 1 = I[\mathcal{C}] \]

in which $L_n$ is the linear graph with $n \geq 0$ vertices and all edges having color $c$ for some $c \in \mathcal{C}$.

2. Suppose $G$ is the $\mathcal{C}$-colored pasting scheme $\text{Tree}_{\mathcal{C}}^{\mathcal{Q}}$ of wheeled trees (for $\mathcal{C}$-colored wheeled operads in $\mathcal{M}$) or $\text{Gr}_{\mathcal{C}}^{\mathcal{Q}}$ of connected wheeled graphs (for
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C-colored wheeled properads in M). Then the initial G-prop has entries

$$I(\ell) = \begin{cases} 1 & \text{if } (\ell) = (\cdot) \text{ for some } c \in C; \\ \bigsqcup_{c \in C} 1_c & \text{if } (\ell) = (c); \\ \emptyset & \text{otherwise.} \end{cases}$$

Here $1_c$ is a copy of the monoidal unit indexed by the color $c$. The only non-trivial structure maps in the initial G-prop are:

- the ones in (3.1.14);
- the $c$-colored contractions

$$I(\cdot) = 1_c \xrightarrow{\text{inclusion}} \bigsqcup_{c \in C} 1_c = I(c)$$

for $c \in C$;
- a map in (3.1.14) for $c \in C$ followed by a $c$-colored contraction.

3.2. Commutative Segments

To define the Boardman-Vogt construction of a generalized prop, we will equip the ordinary internal edges in graphs with a suitable length using the following concept from [BM06] (Def. 4.1).

**Definition 3.2.1.** A segment in $M$ is a tuple

$$(J, \mu, 0, 1, \epsilon)$$

in which:

- $(J, \mu, 0)$ is a monoid in $M$ with multiplication $\mu$, neutral element $0$, absorbing element $1$, and counit $\epsilon$.
- The composite

$$I \sqcup I \xrightarrow{(0, 1)} J \xrightarrow{\epsilon} I$$

is the fold map, i.e., $\epsilon 0 = \epsilon 1 = \text{Id}_1$.

A commutative segment is a segment whose multiplication $\mu$ is commutative.

**Example 3.2.2.** Here are some basic examples of commutative segments.

1. In the category $\text{Top}$ of compactly generated topological spaces, the unit interval $[0, 1]$ equipped with the multiplication

$$a \ast b = \max\{a, b\}$$

is a commutative segment with neutral element $0$ and absorbing element $1$. Alternatively, one can define

$$a \ast b = 1 - (1 - a)(1 - b).$$

2. Similarly, in the category $\text{SSet}$ of simplicial sets, the representable simplicial set $\Delta^1$ is a commutative segment.

3. In the category $\text{Ch}(k)$ of non-negatively graded or unbounded chain complexes over a field $k$ of characteristic $0$, the normalized chain complex $N\Delta^1$ of $\Delta^1$ is a commutative segment.

4. The category $\text{SMod}(k)$ of simplicial $k$-modules has a commutative segment corresponding to $N\Delta^1$ via the Dold-Kan correspondence.
3.2. COMMUTATIVE SEGMENTS

(5) In the category $\mathbf{Cat}$ of all small categories, the category

$$J = \left\{ 0 \xrightarrow{\sim} 1 \right\}$$

with two objects and a unique isomorphism from 0 to 1 is a commutative segment.

MOTIVATION 3.2.3. Given a pasting scheme $G$ and a commutative segment $J$ in $M$, we are about to define a functor with the same symbol

$$J : \mathcal{G}(\frac{\mathcal{C}}{2})^{\text{op}} \longrightarrow M,$$

where $\mathcal{G}(\frac{\mathcal{C}}{2})$ is a substitution category for the pasting scheme $G$. To motivate its definition, consider the category $M = \mathbf{Top}$ and the unit interval $J = [0, 1]$ with multiplication $a \ast b$ given by either $1 - (1 - a)(1 - b)$ or $\max\{a, b\}$. Consider the one-colored connected wheeled graph $G$ whose internal edges are assigned lengths $a, b \in [0, 1]$ as shown below.

Suppose $H_x$ is the exceptional edge, which can be substituted into the vertex $x$ in $G$, and $H_y$ has the same profile as the vertex $y$ in $G$ (i.e., has two inputs and two outputs). Since

$$(H_x, H_y) : K = G(H_x, H_y) \longrightarrow G$$

is a map in the substitution category $\mathcal{G}(\frac{\mathcal{C}}{2})$, we need to define a map $J[G] \longrightarrow J[K]$. As the above picture suggests, for the two internal edges in $G$ connected by the exceptional edge $H_x$, their lengths are multiplied in $K$. The two ordinary internal edges in $H_y$, which become ordinary internal edges in $K$, are assigned length 0 in $K$. This is the essence of the functor $J$ in the next definition.

DEFINITION 3.2.4. Suppose $\mathcal{G}$ is a $\mathcal{C}$-colored pasting scheme, $(\frac{\mathcal{C}}{2})$ is a pair of $\mathcal{C}$-profiles, and $J$ is a commutative segment in $M$. Define a functor

$$J : \mathcal{G}(\frac{\mathcal{C}}{2})^{\text{op}} \longrightarrow M$$

by setting

$$J[G] = \bigotimes_{e \in |G|} J = J^{\otimes |G|}$$

in which $e \in |G|$ means $e$ runs through the set $|G|$ of ordinary internal edges in $G$. For a map $(H_v) : G(H_v) \longrightarrow G \in \mathcal{G}(\frac{\mathcal{C}}{2})$, the required map

$$J[G] \longrightarrow J[G(H_v)]$$

is induced by:
• $0 : \mathbb{1} \to J$ for each ordinary internal edge in each $H_v$ (which must become an ordinary internal edge in $G(H_v)$);

• the multiplication $J \otimes J \xrightarrow{\mu} J$ for each exceptional edge $\uparrow$ connected component in each $H_v$ that connects two distinct ordinary internal edges in $G$;

• the counit $J \xrightarrow{\epsilon} 1$ for each exceptional edge $\uparrow$ connected component in each $H_v$ that connects either (i) one ordinary internal edge and one leg in $G$ or (ii) the two flags of a connected component in $G$ of the form

$$\xi_1^1 c = \begin{array}{c}
v \\
onumber \bigcirc\end{array}
$$

for some $c \in \mathcal{C}$ (Example 1.2.20);

• the identity of $1$ for each exceptional edge $\uparrow$ connected component in each $H_v$ that connects two legs in $G$ and for each exceptional wheel $Q$ connected component in each $H_v$.

**Example 3.2.5.** Consider one-colored unital linear graphs and the map $(H_v) : L_5 \to L_3 \in \mathbb{Ulin}$ in (3.1.4). The corresponding map of the functor $J$ is

$$J[L_3] = J^\otimes 2 \simeq \mathbb{1} \otimes J^\otimes 2 \otimes \mathbb{1} \simeq (0,0,0) \xrightarrow{\otimes 4} J^\otimes 4 = J[L_5]$$

in which all the tensor products are unordered. For instance, in the category $\mathbf{Top}$ with the unit interval $J = [0,1]$ equipped with the maximum operation, this map is given by

$$(a,b) \mapsto (0,\max\{a,b\},0,0)$$

for $a,b \in [0,1]$.

**Example 3.2.6.** Recall the setting of Example 3.1.5 and the map (3.1.6) in the substitution category

$$K = G(H_u,H_v,H_w) \xrightarrow{(H_u,H_v,H_w)} G \in \mathbb{Gr}_{^2}(\mathcal{G})$$

The corresponding map of the functor $J$ is the unordered tensor product

$$(3.2.7) \quad J[G] = J_b \otimes J_c \otimes J_d \otimes \mathbb{1}^\otimes 3 \xrightarrow{\operatorname{id} \otimes \mu \otimes 0 \otimes 0} J_b \otimes J_c \otimes J_d \otimes J_e \otimes J_f = J[K]$$

in which we write $J_b$ for the copy of $J$ corresponding to the internal edge in $G$ or in $K$ with color $b$, and likewise for the other $J_i$. For instance, in the category $\mathbf{Top}$ with the unit interval $J = [0,1]$ equipped with the maximum operation, the above map is given by

$$(3.2.8) \quad (x,y,z) \mapsto (x,\max\{y,z\},0,0,0)$$

for $x,y,z \in [0,1]$.

**Example 3.2.9.** Recall the setting of Example 3.1.8 Then the map

$$J[G] \to J[K]$$
3.3. Coends and their Properties

The purpose of this section is to establish some categorical properties of coends, especially its interactions with a symmetric monoidal product. The reader may consult [Mac98] XI.6 and [Lor∞] for basic information and examples of coends. For the reader’s convenience, we first recall the definition of a coend.

**Definition 3.3.1.** Consider a bifunctor

\[ F: C^{op} \times C \rightarrow M. \]

(1) A **wedge** of \( F \) is a pair \((X, \zeta)\) consisting of

- an object \( X \in M \) and
- maps \( \zeta_c : F(c, c) \rightarrow X \) for \( c \in C \)

such that the diagram

\[
\begin{array}{ccc}
F(d, c) & \xrightarrow{F(g, c)} & F(d, d) \\
\downarrow & & \downarrow \zeta_d \\
F(c, c) & \xrightarrow{\zeta_c} & X
\end{array}
\]

is commutative for each map \( g : c \rightarrow d \in C \).

(2) A **coend** of \( F \) is an initial wedge

\[
\left( \int c \in C F(c, c), \omega \right).
\]

In other words, it is a wedge of \( F \) such that given any wedge \((X, \zeta)\) of \( F \), there exists a unique arrow

\[ h : \int c \in C F(c, c) \rightarrow X \]

in \( M \) such that the diagram

\[
\begin{array}{ccc}
F(c, c) & \xrightarrow{\omega_c} & \int c \in C F(c, c) \\
\downarrow \zeta_c & & \downarrow h \\
& & X
\end{array}
\]

is commutative for each object \( c \in C \).
Since we are assuming that $M$ is cocomplete, every such bifunctor $F$ with $C$ a small category admits a coend, which is unique up to a unique isomorphism. We will also call the object $\int^{c \in C} F(c, c)$ a coend of $F$. A simple exercise shows that we can compute a coend more concretely as a coequalizer as follows.

**Proposition 3.3.2.** Given a bifunctor $F : C^{\text{op}} \times C \to M$ with $C$ a small category, a coend of $F$ exists and is given by a coequalizer

$$\int^{c \in C} F(c, c) = \text{coequal} \left( \bigsqcup_{g \in \text{Mor}(C)} F(d, c) \xrightarrow{i_{c} \cdot F(g, c)} \bigsqcup_{c \in C} F(c, c) \right)$$

in which $g : c \to d$ runs through the morphisms in $C$ and

$$i_{c} : F(c, c) \to \bigsqcup_{c \in C} F(c, c)$$

is the natural inclusion. The natural map $\omega_{c}$ is the composite

$$F(c, c) \xrightarrow{\omega_{c}} \int^{c \in C} F(c, c) \xrightarrow{i_{c}} \bigsqcup_{c \in C} F(c, c)$$

for each object $c \in C$.

**Example 3.3.3.** Probably the simplest example of a coend is the tensor product of modules. Suppose $R$ is a unital associative ring, regarded as a one-object category with morphisms the elements in $R$ and composition the multiplication of elements in $R$. A left $R$-module $N$ is equivalent to a functor $N : R \to \text{Ab}$, the category of abelian groups, sending each element $r \in R$ to the scalar multiplication map $r(-)$. A right $R$-module $M$ is equivalent to a functor $M : R^{\text{op}} \to \text{Ab}$ sending each element $r \in R$ to the scalar multiplication map $(-)r$. Then the tensor product over $R$ of these modules is the coend

$$M \otimes_{R} N = \int^{R} M \otimes N$$

computed in $\text{Ab}$. The tensor product $M \otimes_{R} N$ is usually described as the quotient of the abelian group $M \otimes N$ modulo the relations

$$mr \otimes n \sim m \otimes rn$$

for $r \in R$, $m \in M$, and $n \in N$. The coend description neatly packages all of these relations.

The rest of this section is for Mark’s stuff about coends.

### 3.4. Defining the Boardman-Vogt Construction

**Motivation 3.4.1.** Consider a bifunctor

$$F : C^{\text{op}} \times C \to D$$

with $C$ small and $D$ cocomplete. The coend $\int^{c \in C} F(c, c)$ may be computed as the coequalizer of the parallel maps

$$\bigsqcup_{c, c' \in C} \bigsqcup_{f \in C(c, c')} F(c', c) \xrightarrow{i_{c} \cdot F(c', c)} \bigsqcup_{c \in C} F(c, c)$$
in $D$, in which the top (resp., bottom) map is induced by $F(c', f)$ (resp., $F(f, c)$). Set theoretically the coend of $F$ is a quotient of the sum of the objects $F(c, c)$ as $c$ runs over $C$, with identification coming from the left and the right variables of $F$. The topological W-construction $WO$ in (2.3.5) admits a similar quotient description. This suggests that the $W$-construction of a generalized prop may be defined as a coend over a suitable category of graphs. The next key definition makes this heuristic precise.

**Definition 3.4.2.** Suppose $G$ is a $C$-colored pasting scheme, $(\tilde{G})$ is a pair of $C$-profiles, $P$ is a $G$-prop in $M$, and $J$ is a commutative segment in $M$. Define the coend [Mac98] (IX.6)

$$W(G, J, P)(\tilde{G}) = \int^{G \in G} \tilde{G}(J) \otimes P[J]$$

in which on the right-hand side:

- $P : G(\tilde{G}) \rightarrow M$ is the functor in Def. 3.1.2.
- $J : G(\tilde{G})^{op} \rightarrow M$ is the functor in Def. 3.2.4.

The following observation will be used repeatedly below to check that maps out of the $W$-construction are well-defined or that they have certain properties.

**Lemma 3.4.4.** The object $W(G, J, P)(\tilde{G})$ is uniquely characterized by the following universal property: For each object $G \in G(\tilde{G})$ there is a map

$$J[G] \otimes P[G] \xrightarrow{\omega_{G}} W(G, J, P)(\tilde{G}),$$

and for each map $(H_v) : G(H_v) \rightarrow G \in G(\tilde{G})$ there is a commutative diagram

$$\begin{array}{c}
J[G] \otimes P[G(H_v)] \\
\downarrow (J, id)
\end{array} \xrightarrow{(\triangleright, \otimes_{G})} \begin{array}{c}
J[G] \otimes P[G(H_v)] \\
\downarrow \omega_{G}
\end{array} \xrightarrow{\omega_{G}} W(G, J, P)(\tilde{G}).$$

Furthermore, $W(G, J, P)$ is initial with respect to the above properties.

**Proof.** This is simply the definition of the coend (3.4.3). □

**Example 3.4.6.** Recall the setting of Example 3.1.3 with $ULin$ the one-colored pasting scheme of unital linear graphs and with monoids in $M$ as $ULin$-props. There is only one substitution category $ULin$, which is isomorphic to the opposite category of the wide subcategory of $\Delta$ of functors that preserve both the initial and the terminal objects.

Suppose $(P, \lambda, \eta)$ is a monoid in $M$. Then its $W$-construction (3.4.3) is the coend

$$WP = \int^{L_0 \in ULin} J[L_0] \otimes P[L_0]$$

in which $L_0 = \uparrow$ and $L_n$ for $n \geq 1$ is the one-colored linear graph with $n$ vertices and $n - 1$ internal edges.

$$L_n = [n] = \begin{array}{c}
\downarrow 6 \downarrow 1 \downarrow 2 \downarrow \ldots \downarrow n - 1 \downarrow n
\end{array}$$
So
\[ J[L_n] \otimes P[L_n] = \begin{cases} 1 & \text{if } n = 0; \\ J^n \otimes P^n & \text{if } n \geq 1. \end{cases} \]

For the map
\[ (H_v) : L_5 = L_{3}(H_v) \longrightarrow L_3 \in \mathcal{ULin} \]
in (3.1.4) with
\[ H_1 = L_2, \quad H_2 = \uparrow, \quad \text{and} \quad H_3 = L_3, \]
the commutative square in Lemma 3.4.4 becomes
\[
\begin{array}{ccc}
J \otimes P \otimes J & \overset{\text{Id} \otimes \lambda \otimes \eta \otimes \lambda^2}{\longrightarrow} & J \otimes P \\
\downarrow J \otimes P \otimes \text{Id} & & \downarrow \omega \\
J \otimes P & \overset{\omega \lambda}{\longrightarrow} & WP.
\end{array}
\]
In the top horizontal map, \( \eta : 1 \longrightarrow P \) is the unit in \( P \), and \( \lambda^2 : P^\otimes 3 \longrightarrow P \) is an iteration of the multiplication \( \lambda \) in \( P \). In the left vertical map, \( 0 : 1 \longrightarrow J \) is the 0-end, and \( \mu : J \otimes J \longrightarrow J \) is the multiplication in the segment.

**Example 3.4.7 (Recovering the Classical Boardman-Vogt Construction).** Consider the \( \mathcal{C} \)-colored pasting scheme \( \mathcal{G} = \text{UTree} \) of trees, so \( \text{UTree} \)-props are \( \mathcal{C} \)-colored operads. Our coend definition of the \( W \)-construction (3.4.3) agrees entrywise with the original Boardman-Vogt construction of a topological operad \( P \) (2.3.5). This is an exercise in interpreting the coend in (3.4.3).

Indeed, working topologically suppose the unit interval \( J = [0,1] \) is the commutative segment, and \( P \) is a \( \mathcal{C} \)-colored topological operad. Writing
\[ \mathcal{C} = \text{UTree}^{(c)} \]
for the substitution category for a typical pair of \( \mathcal{C} \)-profiles \( (c) \), by Lemma 3.4.4 the coend
\[
W(\text{UTree}, J, P)^{(c)} = \int_{\mathcal{C}} J[G] \times P[G]
\]
may be computed as the coequalizer of the parallel maps
\[
(3.4.8) \quad \bigg\|_{K \in \mathcal{C}} \bigg\|_{(H_v) \in \mathcal{C}(K,G)} J[G] \times P[K] \xrightarrow{J} \bigg\|_{P \in \mathcal{C}} \bigg\|_{G \in \mathcal{C}} J[G] \times P[G].
\]
This coequalizer in turn may be computed as a quotient
\[
\left( \bigg\|_{G \in \mathcal{C}} J[G] \times P[G] \right)/\sim
\]
in which the identification comes from the parallel maps in (3.4.8).

To see that the identifications in the above coequalizer are the same as those in the classical case (2.3.5), observe that there are three kinds of objects in \( \mathcal{C} = \text{UTree}^{(c)} \):

1. permuted corollas (i.e., unital trees with one vertex),
2. exceptional edges \( \uparrow \), and
3. unital trees with at least two vertices.
3.4. DEFINING THE BOARDMAN-VOGT CONSTRUCTION

Since graph substitution is associative and unital, the above identifications are generated by maps

\[(H_v) : K = G(H_v) \rightarrow G \in \mathcal{C}\]

such that all but one \(H_u\) are corollas. So let us assume \(H_u\) is a corolla for each vertex \(u\) in \(G\) except for one vertex \(v\).

1. Suppose \(H_v\) is a permuted corolla. Then \(K\) is obtained from \(G\) by re-ordering the incoming flags at the vertex \(v\). Since internal edges and legs are not affected, the functor \(J\) does nothing. The operad structure map \(\gamma^{\mathcal{P}}_{H_v}\) of \(\mathcal{P}\) corresponding to a permuted corolla is simply an equivariant structure map of \(\mathcal{P}\). So the corresponding identification on the coproduct \(\bigsqcup_{G \in \mathcal{C}} J[G] \times \mathcal{P}[G]\) is the equivariant identification in (2.3.2).

2. Suppose \(H_v = \uparrow\) is an exceptional edge with color \(c \in \mathcal{C}\). Then \(K = \mathcal{G}(\uparrow)\) is obtained from \(G\) by removing the vertex \(v\) with one incoming flag and one outgoing flag with the same color \(c\) and connecting those two flags. The map

\[J[G] \rightarrow J[G(\uparrow)]\]

is induced by either

- the multiplication \(J^2 \rightarrow J\) (if \(H_v\) connects two internal edges) or
- the counit \(J \rightarrow *\) (if \(H_v\) connects a leg and some other flag).

The structure map

\[\gamma^{\mathcal{P}}_{\uparrow} : \mathcal{P}[\uparrow] = * \rightarrow \mathcal{P}(\uparrow)\]

is the \(c\)-colored unit of \(\mathcal{P}\). The identification on the coproduct \(\bigsqcup_{G \in \mathcal{C}} J[G] \times \mathcal{P}[G]\) provided by the maps

\[J[G] \times \mathcal{P}[G(\uparrow)] \xrightarrow{J} J[G(\uparrow)] \times \mathcal{P}[G(\uparrow)]\]

is exactly the unit identification in (2.3.3).

3. Suppose \(H_v\) is a unital tree with at least two vertices, so it has at least one internal edge. Then \(K = \mathcal{G}(H_v)\) is obtained from \(G\) by replacing the vertex \(v\) with the unital tree \(H_v\). In the diagram

\[J[G] \times \mathcal{P}[K] \xrightarrow{J} J[K] \times \mathcal{P}[K]\]

the map \(J\) is the product of the identity map and copies of the inclusion \(0 \rightarrow J\) indexed by the internal edges in \(H_v\). The map \(\mathcal{P}\) is the product of the identity map and the structure map \(\gamma^{\mathcal{P}}_{H_v}\), which is an iteration of the various \(\circ_{i}\)-compositions in \(\mathcal{P}\). The corresponding identification on the
coproduct $\coprod_{G \in \mathcal{C}} J[G] \times P[G]$ is an iteration of the compositional identification in \eqref{eq:coend-defn}. Therefore, even in the classical case, with the use of the substitution categories, our coend definition \eqref{eq:coend-defn} gives a conceptual and easy way to package the $W$-construction of an operad.

**Example 3.4.9.** Consider the setting of Example \ref{exm:3.1.5} with $P$ a $\mathcal{C}$-colored properad in $M$. For any pair of $\mathcal{C}$-profiles $(\xi, \eta)$, the $(\xi, \eta)$-entry of the $W$-construction is the object

$$W(\mathcal{G}, J, P)(\xi, \eta) = \int_{G \in \mathcal{G}} \mathcal{G}^{\xi}(G) \times J[G] \otimes P[G].$$

In particular, for the map \eqref{eq:3.1.6} in the substitution category $K = G(H_u, H_v, H_w) \xrightarrow{(H_u, H_v, H_w)} G \in \mathcal{G}^{\eta}(\eta)$, the corresponding commutative diagram in Lemma \ref{lem:3.4.4} takes the form

$$
\begin{array}{ccc}
J[G] \otimes P[K] & \xrightarrow{id_{J[G]} \otimes p \otimes p \otimes p \otimes id_{P[K]}} & J[G] \otimes P[G] \\
\downarrow & & \downarrow \omega_G \\
J[K] \otimes P[K] & \xrightarrow{\omega_K} & W(\mathcal{G}, J, P)(\xi, \eta).
\end{array}
$$

The top horizontal map is the tensor product of the identity map of $J[G]$ and the map $P[K] \rightarrow P[G]$ in \eqref{eq:3.1.7}. The left vertical map is the tensor product of the map $J[G] \rightarrow J[K]$ in \eqref{eq:3.2.7} and the identity map of $P[K]$.

**Example 3.4.10.** Consider the setting of Example \ref{exm:3.1.8} with $P$ a $\mathcal{C}$-colored wheeled properad in $M$. For any pair of $\mathcal{C}$-profiles $(\xi, \eta)$, the $(\xi, \eta)$-entry of the $W$-construction is the object

$$W(\mathcal{G}, J, P)(\xi, \eta) = \int_{G \in \mathcal{G}} \mathcal{G}^{\xi}(G) \times J[G] \otimes P[G].$$

In particular, for the map \eqref{eq:3.1.9} in the substitution category $K = G(H_u, H_v) \xrightarrow{(H_u, H_v)} G \in \mathcal{G}^{\eta}(\eta)$, the corresponding commutative diagram in Lemma \ref{lem:3.4.4} takes the form

$$
\begin{array}{ccc}
J[G] \otimes P[K] & \xrightarrow{(\mu, 0 \otimes 3) \otimes id_{P[K]}} & J[G] \otimes P[G] \\
\downarrow & & \downarrow \omega_G \\
J[K] \otimes P[K] & \xrightarrow{\omega_K} & W(\mathcal{G}, J, P)(\xi, \eta).
\end{array}
$$

The top horizontal map is the tensor product of the identity map of $J[G]$ and the map in \eqref{eq:3.1.10}. The left vertical map is the tensor product of the product in \eqref{eq:3.2.10} and the identity map of $P[K]$.

**Example 3.4.11.** Consider the setting of Example \ref{exm:3.1.11} with $\mathcal{G}$ a $\mathcal{C}$-colored pasting scheme and $\text{Com}^{\mathcal{G}}$ the $\mathcal{G}$-prop in $M$ in which every entry is the monoidal unit. Since every

$$\text{Com}^{\mathcal{G}}[G] \cong 1$$
with $\gamma^{\text{Com}}_G$ the identity map, the commutative diagram in Lemma 3.4.4 is isomorphic to the diagram

$$
\begin{array}{ccc}
J[G] & \xrightarrow{=} & J[G] \\
\downarrow J & & \downarrow \omega_G \\
J[G(H_v)] & \xrightarrow{\omega_G(H_v)} & W(\mathcal{G}, J, \text{Com}^{\mathcal{G}})(\check{v})
\end{array}
$$

This means that each ($\check{v}$)-entry of the $W$-construction

$$
W(\mathcal{G}, J, \text{Com}^{\mathcal{G}})(\check{v}) = \int_{G \in \mathcal{G}} J[G] \otimes \text{Com}^{\mathcal{G}}[G]
$$

is a colimit of the functor $J : \mathcal{G}(\check{v})^{\text{op}} \to M$.

### 3.5. Generalized Prop Structure

We now define the $G$-prop structure on the $W$-construction

**Convention 3.5.1.** We will often abbreviate $J[G] \otimes P[G]$ to $(J \otimes P)[G]$.

**Motivation 3.5.2.** As discussed before, for a $\mathcal{G}$-prop $P$ we think of the $W$-construction $W(\mathcal{G}, J, P)$ as the space of decorated graphs $(J \otimes P)[G]$, with vertices decorated by the $\mathcal{G}$-prop $P$ and ordinary internal edges decorated by the commutative segment $J$. For example, if $G$ is the graph on the left,

![Graph](image)

then the decorated graph

$$(J \otimes P)[G] = J[G] \otimes \bigotimes_{w \in G} P(w)$$

is depicted on the right.

There are certain identifications among these decorated graphs parametrized by maps in the substitution categories. In the classical Boardman-Vogt construction, the operad structure is given by grafting of decorated trees, with new internal edges given length 1. In our setting, grafting is subsumed by the substitution categories, and the concept of length is given by $J$. This motivates the following definition of the $G$-prop structure of the $W$-construction.

**Definition 3.5.3.** In the context of Def. 3.4.2 suppose $G \in \mathcal{G}(\check{v})$. Then

$$W(\mathcal{G}, J, P)[G] \cong \bigotimes_{v \in G} W(\mathcal{G}, J, P)(v)$$

is defined by

$$\bigotimes_{v \in G} W(\mathcal{G}, J, P)(v) = \bigotimes_{v \in G} \bigotimes_{v \in G} (J \otimes P)[H_v].$$
The structure map

\[ W(G, J, P)[G] \xrightarrow{\gamma_G^{W(G, J, P)}} W(G, J, P)[\mathcal{G}^2] \]

is defined by the maps

\[ \bigotimes_{v \in G} (J \otimes P)[H_v] \xrightarrow{\pi} (J \otimes P)[G(H_v)] \xrightarrow{\omega_G[H_v]} W(G, J, P)[\mathcal{G}^2] \]

in which the restriction of \( \pi \) to the \( P \)-component is the isomorphism

\[ \bigotimes_{v \in G} P[H_v] \cong P[G(H_v)]. \]

The restriction of \( \pi \) to the \( J \)-component is the map

\[ \left( \bigotimes_{v \in G} J[H_v] \right) \otimes \bigotimes_{E} (1 \xrightarrow{} J) \]

in which \( E \) is the set of ordinary internal edges in \( G(H_v) \) that are not in any of the \( H_v \).

**Lemma 3.5.8.** The map \( \gamma_G^{W(G, J, P)} \) in (3.5.5) is well-defined.

**Proof.** For each vertex \( v \in G \), suppose \( H_v \in G(v) \) and, for each vertex \( u \in H_v \), \( D_{vu} \in G(u) \), so there is a map

\[ (D_{vu}) : H_v(D_{vu}) \longrightarrow H_v \in G(v). \]

In the diagram

\[ \bigotimes_{v \in G} (J[H_v] \otimes P[H_v(D_{vu})]) \xrightarrow{\bigotimes_{v} J} \bigotimes_{v \in G} (J \otimes P)[H_v(D_{vu})] \]

the left triangle and the top trapezoid are commutative by inspection, in which \( E \) and \( 1 : 1 \longrightarrow J \) are as in (3.5.7). The lower trapezoid is commutative by Lemma 3.4.4 since

\[ (D_{vu}) : (G(H_v)) (D_{vu}) \longrightarrow G(H_v) \]

is a map in the substitution category \( G^2 \). This shows that the collection of maps (3.5.6) yields a well-defined map \( \gamma_G^{W(G, J, P)} \) from the right-most colimit in (3.5.4) to \( W(G, J, P)[\mathcal{G}^2] \).

**Example 3.5.9.** Recall the setting of Examples 3.1.3 and 3.4.6 with \( \text{ULin} \) the one-colored pasting scheme of unital linear graphs, monoids in \( M \) as \( \text{ULin}-\text{props}, \)
and $\text{ULin}$ the only substitution category. Suppose $P$ is a monoid in $M$. Then the multiplication on the object

$$WP = \int L_n^{e\text{ULin}} J[L_n] \otimes P[L_n]$$

is given by insisting that the diagrams

$$\bigotimes_{i=1}^{n} (J[L_{k_i}] \otimes P[L_{k_i}]) \xrightarrow{\pi} J[L_k] \otimes P[L_k]$$

$$\bigotimes \omega_{L_{k_i}}$$

be commutative for all $n, k_1, \ldots, k_n \geq 0$, where

$$L_k = L_n(L_{k_1}, \ldots, L_{k_n}).$$

For instance, consider $L_3, H_1 = L_2, H_2 = \uparrow, $ and $H_3 = L_3$, so $L_5 = L_3(H_3) = \uparrow$.

Then the diagram

$$\begin{array}{ccc}
(F \otimes P^{\otimes 2}) \otimes 1 \otimes (F^{\otimes 2} \otimes P^{\otimes 3}) & \xrightarrow{1} & J^{\otimes 4} \otimes P^{\otimes 5} \\
\otimes \omega_{H_v} & & \otimes \omega_{L_5} \\
(WP)[L_3] = (WP)^{\otimes 3} & \xrightarrow{\gamma_{L_3}^{WP}} & WP
\end{array}$$

is commutative. The top horizontal map is a permutation of tensor factors (moving all the $P$ factors to the right) followed by the tensor product of the identity map and $1 : 1 \rightarrow J$.

**Example 3.5.10 (Recovering the Operad Structure in the Classical Boardman-Vogt Construction).** Continuing Example [3.4.7] above, consider the category of compactly generated spaces with commutative segment $J = [0, 1]$. The pasting scheme $G = \text{UTree}$ of $C$-colored unital trees has $C$-colored operads as $\text{UTree}$-props. For a $C$-colored topological operad $O$, as explained in Example [3.4.7] our coend definition [3.4.3] and the classical Boardman-Vogt construction [2.3.5] yield the same underlying object. Moreover, our structure map $\gamma_{G,W,J,O}^{WP}$ in [3.5.5] agrees with the operad structure in the classical Boardman-Vogt construction, discussed between Examples [2.3.6] and [2.3.7].

For instance, in Examples [2.3.6] and [2.3.7] the operadic comp-2 operation in [2.3.8] is the map

$$(WO)[T] = (WO)(\delta)^{d}(a,b,c,d) \times (WO)(\delta)^{b}(c,d) \xrightarrow{v_2 = \gamma_{T}^{W,J,O}} (WO)(\delta)^{d}(a,b,c,d).$$
in which $WO = W(G, J, O)$ and $T$ is the tree:

![Tree Diagram]

\[(3.5.11)\]

Applied to $A \in (WO)_{a,b,d,c}^d$ and $B \in (WO)_{a,b}^b$, where

we see from the definition (3.5.6) that $\gamma^W_{T,G,J}(A, B)$ is the decorated tree $T(A, B)$ with the new internal edge (i.e., the internal edge in $T$) given length 1. This is the same as $A \circ_2 B$ in (2.3.9).

**Example 3.5.12.** Consider the setting of Example 3.1.5 with $P$ a $\mathcal{C}$-colored properad in $M$ and

\[
\begin{align*}
(J \otimes P)[H_w] &= J_e \otimes J_f \otimes P(w_1) \otimes P(w_2); \\
(J \otimes P)[H_u] &= J_d \otimes P(u_1) \otimes P(u_2); \\
(J \otimes P)[H_v] &= 1; \\
(J \otimes P)[K] &= J_b \otimes J_c \otimes J_d \otimes J_e \otimes J_f \otimes P(w_1) \otimes P(w_2) \otimes P(u_1) \otimes P(u_2).
\end{align*}
\]

As before we write $J_b$ for the copy of $J$ corresponding to an internal edge with color $b$, and likewise for the other $J_f$. The definition of $\gamma^W_{G,J,P}$ (3.5.6) implies the diagram

\[
\begin{array}{ccc}
(J \otimes P)[H_w] \otimes (J \otimes P)[H_u] \otimes (J \otimes P)[H_v] & \xrightarrow{\pi} & (J \otimes P)[K] \\
\omega_{H_w} \otimes \omega_{H_u} \otimes \omega_{H_v} & & \omega_K \\
(WP)[G] = (WP)(w) \otimes (WP)(u) \otimes (WP)(v) & \xrightarrow{\gamma^WP_G} & (WP)_c^c
\end{array}
\]

is commutative, in which $WP = W(G, J, P)$. The top horizontal map $\pi$ is induced by the map

\[
1 \otimes 1 : 1 \otimes 1 \longrightarrow J_b \otimes J_c.
\]
Example 3.5.13. Consider the setting of Example 3.1.8 with \( P \) a \( C \)-colored wheeled properad in \( M \) and

\[
\begin{align*}
(J \otimes P)[H_a] &= J_a \otimes J_b \otimes J_d \otimes P(x) \otimes P(y); \\
(J \otimes P)[H_b] &= 1; \\
(J \otimes P)[K] &= J_a \otimes J_b \otimes J_c \otimes J_d \otimes P(x) \otimes P(y).
\end{align*}
\]

The definition of \( \gamma^{W(G,J,P)}_G \) (3.5.6) implies the diagram

\[
\begin{array}{ccc}
(J \otimes P)[H_a] \otimes (J \otimes P)[H_a] & \xrightarrow{\pi} & (J \otimes P)[K] \\
\downarrow \omega_{H_a} & & \downarrow \omega_K \\
(WP)[G] = (WP)(u) \otimes (WP)(v) & \xrightarrow{\gamma^{W,P}_G} & (WP)^{(G)}
\end{array}
\]

is commutative, in which \( WP = W(G,J,P) \). The top horizontal map \( \pi \) is induced by the map \( 1 : 1 \to J_c \).

Example 3.5.14. Consider the setting of Examples 3.1.11 and 3.4.11 with \( G \) a \( C \)-colored pasting scheme and \( \text{Com}^G \) the \( G \)-prop with each entry the monoidal unit in \( M \) and each structure map the identity map. Each \( (\frac{G}{d}) \)-entry of its \( W \)-construction is a colimit

\[
W(G,J,\text{Com}^G)(\frac{G}{d}) = \text{colim}( \text{Com}^G_{(\frac{G}{d})} \xrightarrow{J} M )
\]

of the functor \( J \). The \( G \)-prop structure map of the \( W \)-construction is defined by the commutative diagram

\[
\begin{array}{ccc}
\bigotimes_{v \in G} J[H_v] & \xrightarrow{\pi} & J[G(H_v)] \\
\downarrow \bigotimes_{v \in G} \omega_{H_v} & & \downarrow \omega_{G(H_v)} \\
\bigotimes_{v \in G} W(G,J,\text{Com}^G)(v) & \xrightarrow{} & W(G,J,\text{Com}^G)(\frac{G}{d})
\end{array}
\]

for each \( G \in \mathcal{G}(\frac{G}{d}) \) and each choice of \( H_v \in \mathcal{G}(v) \) for \( v \in G \). Here \( \pi \) is the map in (3.5.6), which is induced by copies of the map \( 1 : 1 \to J_c \).

The following observation shows that \( \gamma^{W(G,J,P)}_G \) is associative with respect to graph substitution.

Motivation 3.5.15. Thinking of the \( W \)-construction \( W(G,J,P) \) of a \( G \)-prop \( P \) as the space of decorated graphs \( (J \otimes P)[G] \) with some identifications, we defined the structure map \( \gamma^{W(G,J,P)}_G \) via graph substitution, giving new ordinary internal edges length 1. This process of giving new internal edges length 1 is associative, as can be seen in the following example.
The graph substitution
\[ G(H(D_x, D_y)) = (G(H))(D_x, D_y) \]
is defined, and \( D_x \) and \( D_y \) are decorated. If we first substitution \((J \otimes P)[D_x]\) and \((J \otimes P)[D_y]\) into \( H \), then the two internal edges in \( H \) are given length 1 in \( H(D_x, D_y) \). When this is substituted into \( G \), the internal edge in \( G \) is given length 1, resulting in three internal edges with length 1 in \( G(H(D_x, D_y)) \). Similarly, \( G(H) \) has three internal edges. When we substitute \((J \otimes P)[D_x]\) and \((J \otimes P)[D_y]\) into \( G(H) \), the three internal edges of the latter are given length 1. So either way we obtain the same result. This is formally proved in the next observation.

**Lemma 3.5.16.** In the context of Def. 3.5.3, the diagram
\[ W(G, J, P)[G(H_v)] \xrightarrow{\gamma_{G(H_v)}} W(G, J, P)(\gamma_{H_v}) \]
\[ \cong \bigoplus_{v \in G} W(G, J, P)[H_v] \xrightarrow{\bigoplus_{v \in G} \gamma_{H_v}} \bigoplus_{v \in G} W(G, J, P)(v) = W(G, J, P)[G] \]
is commutative.

**Proof.** To simplify the notation, we will write \( \gamma_{W(G, J, P)} \) as \( \gamma \). Using the notations in the proof of Lemma 3.5.8 similar to (3.5.4), we have
\[ W(G, J, P)[G(H_v)] \cong \colim_{v \in G} \prod_{u \in H_v} \mathcal{G}(u) \left( \bigotimes_{v \in G, u \in H_v} (J \otimes P)[D_{vu}] \right) \]
The maps \( \gamma_{G(H_v)} \), \( \gamma_G \), and \( \gamma_{H_v} \) are determined by the map \( \pi \) in (3.5.6), which does nothing in the P-component. In the diagram in the statement of the Lemma, both composites in the \( J \)-component are the identity map of \( \bigotimes_{v \in G} J[D_{vu}] \) tensored with copies of the map \( 1 \xrightarrow{1} J \) indexed by the set of ordinary internal edges in \( G(H_v)(D_{vu}) \) that are not in any of the \( D_{vu} \).
Theorem 3.5.17. Suppose $\mathcal{G}$ is a $\mathcal{C}$-colored pasting scheme, $\mathcal{P}$ is a $\mathcal{G}$-prop in $\mathcal{M}$, and $J$ is a commutative segment in $\mathcal{M}$. Then $W(\mathcal{G}, J, \mathcal{P})$ is a $\mathcal{G}$-prop in $\mathcal{M}$ with structure maps $\gamma^{W(\mathcal{G}, J, \mathcal{P})}$ (3.5.5).

Proof. Associativity of $\gamma^{W(\mathcal{G}, J, \mathcal{P})}$ is proved in Lemma 3.5.16. For unity observe that for a corolla $C$, the structure map $\gamma^{W(\mathcal{G}, J, \mathcal{P})} C$ is the identity map because $C$ has a unique vertex and $C(H) = H$ whenever the graph substitution $C(H)$ makes sense. \qed

Remark 3.5.18. When $\mathcal{G}$ is the pasting scheme of $\mathcal{C}$-colored unital trees, $\mathcal{G}$-props are $\mathcal{C}$-colored operads. When the underlying category $\mathcal{M}$ is the category of compactly generated spaces, our $W(\mathcal{G}, J, \mathcal{P})$ agrees with the one in [BV72] (III.1) for topological colored operads. For a general symmetric monoidal category with a commutative segment, our $W(\mathcal{G}, J, \mathcal{P})$ also agrees with the one in [BM06], which deals with one-colored operads. However, in [BM06] the $W$-construction is defined as a sequential colimit of pushouts. In contrast, our $W$-construction has a simpler description, namely, a coend over the substitution category of the functor $J \otimes \mathcal{P}$. A suitable filtration of our construction arises as a consequence of filtering the substitution category, as we will show in Section 5.

Example 3.5.19. Consider the setting of Examples 3.1.3, 3.4.6, and 3.5.9 with $\mathcal{ULin}$ the one-colored pasting scheme of unital linear graphs, monoids in $\mathcal{M}$ as $\mathcal{ULin}$-props, and only one substitution category $\mathcal{ULin}$. Suppose $(\mathcal{P}, \lambda, \eta)$ is a monoid in $\mathcal{M}$. Its $W$-construction (3.4.3) is the coend $WP = \int_{L_n \in \mathcal{ULin}} J[L_n] \otimes P[L_n]$, which is a monoid by Theorem 3.5.17.

To illustrate the associativity of the multiplication of WP, consider the linear graph $G = L_2$, into which we substitute $H_1 = L_2 = H_2$, so $G(H_1, H_2) = L_4$. The associativity diagram in Lemma 3.5.16 takes the form

\[
(WP)[L_4] = (WP) \otimes^4 \xrightarrow{\gamma_{L_4}^{WP}} (WP)
\]

\[
\xymatrix{(WP)[L_2] \otimes^2 \ar[r]^-{\gamma_{L_2}^{WP} \otimes \gamma_{L_2}^{WP}} \ar[d]_\cong & (WP) \otimes^2 \ar[d] \\xrightarrow{\gamma_{L_2}^{WP}} (WP)[L_2].}
\]

For instance, suppose in $H_1$ (resp., $H_2$) we substitute in $L_3$ and $L_2$ (resp., $L_4$ and $L_2$). By the associativity of graph substitution, we have

\[
L_2(L_2, L_2)(L_3, L_2, L_4, L_2) = L_2(L_2(L_3, L_2), L_2(L_4, L_2))
\]

\[
= L_2(L_5, L_6) = L_{11}.
\]

This iterated graph substitution decomposition of the linear graph $L_{11}$ is depicted as follows, where gray boxes indicate graph substitution.
Then part of the associativity diagram (3.5.20) says the diagram

\[
\begin{align*}
\left[(J \otimes P \otimes J) \otimes (J \otimes P \otimes J)\right] & \xrightarrow{1 \otimes 1} \left[(J \otimes P \otimes J) \otimes (J \otimes P \otimes J)\right] \\
(J \otimes P \otimes J) & \xrightarrow{1} J[11] \otimes P[11]
\end{align*}
\]

is commutative, where \(1 : 1 \to J\).

**Example 3.5.21.** Consider the setting of Examples 3.1.8, 3.4.10, and 3.5.13 with \(\text{Gr}_{\rho}\) the \(\mathcal{C}\)-colored pasting scheme of connected wheeled graphs. For a \(\mathcal{C}\)-colored wheeled properad \(P\) (i.e., a \(\text{Gr}_{\rho}\)-prop) in \(M\) and a pair of \(\mathcal{C}\)-profiles \((\mathcal{J})\), the \((\mathcal{J})\)-entry of the \(W\)-construction is the coend

\[
W(G, J, P)(\mathcal{J}) = \int^{G \in \text{Gr}_{\rho}} J[G] \otimes P[G].
\]

By Theorem 3.5.17 \(W(G, J, P)\) is a \(\mathcal{C}\)-colored wheeled properad in \(M\).

Using the graphs \(G, H_u, H_v = \uparrow_{c}\), and \(K = G(H_u, H_v)\) in Example 3.1.8, the associativity diagram in Lemma 3.5.16 takes the form

\[
\begin{align*}
(WP)[K] &= (WP)(x) \otimes (WP)(y) \xrightarrow{\gamma^W_K} (WP)(z) \\
&\downarrow z \quad \downarrow \gamma^W_G \\
(WP)[H_u] \otimes (WP)[H_v] &\xrightarrow{\gamma^K_{H_u} \otimes \gamma^K_{H_v}} (WP)(u) \otimes (WP)(v) = (WP)[G]
\end{align*}
\]

in which \(WP = W(G, J, P)\).

For instance, consider the following iterated graph substitution in \(\text{Gr}_{\rho}\), where the gray boxes and the gray arrows indicate graph substitution.

In \(D_x\) (resp., \(D_y\)) the input and output legs are ordered to match those of the vertex \(x\) (resp., \(y\)) in \(H_u\), so the graph substitution \(H_u(D_x, D_y)\) is defined. The
listing at each vertex in \( D_x \) and \( D_y \) is arbitrary. Writing

\[ N = G(H_u, H_v)(D_x, D_y), \]

we have that:

\[
(J \otimes P)[D_x] = J_e \otimes J_f \otimes P(s) \otimes P(t);
\]

\[
(J \otimes P)[D_y] = J_e \otimes J_f \otimes P(q) \otimes P(r);
\]

\[
(J \otimes P)[H_u(D_x, D_y)] = J_a \otimes J_b \otimes J_d \otimes J_e^{(2)} \otimes J_f^{(2)} \otimes P(s) \otimes P(t) \otimes P(q) \otimes P(r);
\]

\[
(J \otimes P)[N] = J_a \otimes J_b \otimes J_c \otimes J_d \otimes J_e^{(2)} \otimes J_f^{(2)} \otimes P(s) \otimes P(t) \otimes P(q) \otimes P(r).
\]

Then the associativity diagram (3.5.22) implies the diagram

\[
\begin{array}{c}
(J \otimes P)[D_x] \otimes (J \otimes P)[D_y] \otimes 1 \longrightarrow (J \otimes P)[N] \\
\downarrow 1 \otimes 1 \\
(J \otimes P)[H_u(D_x, D_y)] \otimes 1
\end{array}
\]

is commutative, in which \( 1 \colon 1 \longrightarrow J \).
Categorical Properties of the Boardman-Vogt Construction

We continue to assume $G$ is a pasting scheme, and $(M, \otimes, 1)$ is a cocomplete symmetric monoidal category whose monoidal product commutes with colimits on both sides. Suppose $(J, \mu, 0, 1, \epsilon)$ is a commutative segment in $M$. In this chapter, we establish some categorical properties of the $W$-construction. In Section 4.1 we observe that there is a natural augmentation $W(G, J, P) \xrightarrow{\eta} P$ for each $G$-prop $P$. It is through this augmentation that we think of the $W$-construction as a resolution of $P$. For nice enough $P$, we will see later that the augmentation is a weak equivalence of $G$-prods.

In Section 4.2 we show that the augmentation $\eta$ provides a factorization of the counit $FUP \xrightarrow{\delta} W(G, J, P) \xrightarrow{\eta} P$ associated to a certain inclusion $G_0 \subset G$ of pasting schemes. The pasting scheme $G_0$ contains only the graphs in $G$ that have no ordinary internal edges. For nice enough $P$, we will see later that the map $\delta$ is a cofibration of $G$-prods, which implies that $W(G, J, P)$ is a cofibrant $G$-prop.

In Section 4.3 and Section 4.4 we establish naturality properties of the $W$-construction $W(G, J, P)$ in all three variables and in the underlying category.

4.1. Augmentation

Motivation 4.1.1. Suppose $P$ is a $G$-prop in $M$ for some pasting scheme $G$, and $U$ is the forgetful functor from $G$-prods to a product of copies of $M$. The counit of the adjunction $F \rightarrow U$ yields a map $FUP \longrightarrow P$ of $G$-prods. The $W$-construction $W(G, J, P)$ is roughly speaking the free $G$-prop construction in which ordinary internal edges are decorated by a chosen commutative segment $J$, together with some identifications. It is therefore natural to expect a similar map $W(G, J, P) \longrightarrow P$ of $G$-prods in which one first forgets the lengths of the ordinary internal edges in the decorated graphs $(J \otimes P)[G]$ in $W(G, J, P)$. 

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For instance, suppose $G \in \mathcal{G}(\mathcal{C})$ and $(J \otimes P)[G]$ is the decorated graph on the left.

After forgetting the lengths of the internal edges, we obtain the middle decorated graph $P[G]$. Using the $\mathcal{G}$-prop structure map $\gamma_P$ of $P$, we can map from $P[G]$ to an entry of $P$. This is made precise in the following observation.

**Proposition 4.1.2.** In the context of Def. 3.4.2, the counit of the segment $\epsilon : J \to 1$ and the $\mathcal{G}$-prop structure maps of $P$ induce a map

$$\eta : W(\mathcal{G}, J, P) \to P$$

of $\mathcal{G}$-props.

**Proof.** For each pair of $\mathcal{C}$-profiles $(\mathcal{C})$, the map

$$\eta : W(\mathcal{G}, J, P)(\mathcal{C}) \to P(\mathcal{C})$$

is induced by the maps

$$(J \otimes P)[G] \xrightarrow{\epsilon \otimes 1} 1 \otimes P[G] \equiv P[G] \xrightarrow{\gamma_P} P(\mathcal{C})$$

for objects $G \in \mathcal{G}(\mathcal{C})$. To see that this collection of maps yields a well-defined map out of $W(\mathcal{G}, J, P)(\mathcal{C})$, suppose $(H_v) : G(H_v) \to G$ is a map in $\mathcal{G}(\mathcal{C})$. In the diagram

the right triangle is commutative by the associativity of the $\mathcal{G}$-prop structure map $\gamma_P$. The left trapezoid is commutative because $\epsilon$ is the counit of $J$, which in particular means the diagrams

are commutative. So we have a well-defined entry map $\eta : W(\mathcal{G}, J, P)(\mathcal{C}) \to P(\mathcal{C})$ for each pair $(\mathcal{C})$. 
4.1. AUGMENTATION

To see that \( \eta \) is a map of \( G \)-props, we must show that the diagram

\[
W(G, J, P)(\mathcal{C}) \xrightarrow{\otimes v \eta} P[G] \\
\downarrow \quad \quad \quad \downarrow \gamma^G \\
W(G, J, P)(\mathcal{C}) \xrightarrow{\eta} P(\mathcal{C})
\]

is commutative for each \( G \in G(\mathcal{C}) \). Using the right-most colimit in (3.5.4), it suffices to show that for each map \( (H_v : G(H_v) \to G) \) in \( G(\mathcal{C}) \), the outside diagram in

\[
\begin{array}{ccc}
\bigotimes_{v \in G} (J \otimes P)[H_v] & \xrightarrow{\otimes v \epsilon[H_v]} & \bigotimes_{v \in G} P[H_v] \\
\downarrow \pi & \quad \quad \quad \quad \quad \quad \downarrow \pi & \quad \quad \quad \quad \quad \quad \downarrow \pi \\
(J \otimes P)[G(H_v)] & \xrightarrow{\epsilon[G(H_v)]} & P[G(H_v)] \\
\end{array}
\]

is commutative. The right square is commutative by the associativity of the \( G \)-prop structure map \( \gamma^P \). The left square is commutative because the composite \( \epsilon \) is the identity map on \( I \).

**Corollary 4.1.4.** For each pair of \( \mathcal{C} \)-profiles \( (\mathcal{C}) \), the composite

\[
P(\mathcal{C}) = (J \otimes P)[C] \xrightarrow{\omega_C} W(G, J, P)(\mathcal{C}) \xrightarrow{\eta} P(\mathcal{C})
\]

is the identity map, where \( C \) is the corolla with profiles \( (\mathcal{C}) \).

**Proof.** By the definition of \( \eta \), this composite is the \( G \)-prop structure map \( \gamma^P \), which is the identity map by the unity of the \( G \)-prop \( P \).

**Example 4.1.5.** Consider the setting of Examples 3.1.3 and 3.4.6 with \( ULin \) the one-colored pasting scheme of unital linear graphs, monoids in \( M \) as \( ULin \)-props, and \( ULin \) as the only substitution category. For a monoid \( P \) in \( M \), its \( W \)-construction

\[
WP = \int^{L_n \in ULin} J[L_n] \otimes P[L_n]
\]

is a monoid in \( M \). The augmentation \( \eta: WP \to P \) is defined by the commutative diagrams

\[
\begin{array}{ccc}
(J \otimes P)[L_n] & \xrightarrow{\epsilon[L_n]} & I \otimes L_n \otimes P \cong P^n \\
\downarrow \omega_{L_n} & \quad \quad \quad \quad \quad \quad \downarrow \text{multiply} & \quad \quad \quad \quad \quad \quad \downarrow \text{multiply} \\
WP & \xrightarrow{\eta} & P
\end{array}
\]

for \( n \geq 0 \). The right vertical map is the unit of \( P \) if \( n = 0 \), the identity map if \( n = 1 \), and the \( (n - 1) \)-fold iterate of the multiplication in \( P \) if \( n \geq 2 \).

**Example 4.1.6.** As in Examples 3.4.7 and 3.5.10 consider the \( \mathcal{C} \)-colored pasting scheme \( \mathcal{G} = UTree \) of unital trees, so \( UTree \)-props are \( \mathcal{C} \)-colored operads in \( M \). For a \( \mathcal{C} \)-colored operad \( P \), the \( W \)-construction is a \( \mathcal{C} \)-colored operad whose \( (\mathcal{C}) \)-entry is the coend

\[
W(G, J, P)(\mathcal{C}) = \int^{T \in UTree} J[T] \otimes P[T].
\]
The augmentation \( \eta \) is defined by the commutative diagrams

\[
\begin{array}{ccc}
(J \otimes P)[T] & \xrightarrow{\epsilon[T]} & 1^{\otimes |T|} \otimes P[T] \\
\omega_T & \downarrow & \downarrow \gamma^P_T \\
W(G, J, P)(\frac{\gamma}{T}) & \xrightarrow{\eta} & P(\frac{\gamma}{T})
\end{array}
\]

for trees \( T \) with profiles \( \frac{T}{\gamma} \). For instance, if \( T \) is the unital tree in \([3.5.11]\), then the right vertical map \( \gamma^P_T \) is the comp-2 operation

\[
P[T] = P(a, b, c, d) \otimes P(b, c) \xrightarrow{\sigma_2} P(a, b, c, d, e)
\]

of the operad \( P \).

**Example 4.1.7.** As in Examples \([3.1.8]\) and \([3.4.10]\), consider the pasting scheme \( \text{Gr}_c^G \) of \( C \)-colored connected wheeled graphs, so \( \text{Gr}_c^G \)-props are \( C \)-colored wheeled properads in \( M \). For a \( C \)-colored wheeled properad \( P \), the \( W \)-construction is a \( C \)-colored wheeled properad whose \( \frac{T}{\gamma} \)-entry is the coend

\[
W(G, J, P)(\frac{T}{\gamma}) = \int_{G \in \text{Gr}_c^G(\frac{T}{\gamma})} J[G] \otimes P[G].
\]

The augmentation \( \eta \) is defined by the commutative diagrams

\[
\begin{array}{ccc}
(J \otimes P)[G] & \xrightarrow{\epsilon[G]} & 1^{\otimes |G|} \otimes P[G] \\
\omega_G & \downarrow & \downarrow \gamma^P_G \\
W(G, J, P)(\frac{T}{\gamma}) & \xrightarrow{\eta} & P(\frac{T}{\gamma})
\end{array}
\]

For instance, consider the graph \( G \) in Example \([3.1.8]\), which may be constructed as the graph substitution of a simply connected graph \( D \) into a contracted corolla \( B \) as follows.

Then the right vertical map \( \gamma^P_G \) in the previous commutative diagram is the composite

\[
P[G] \xrightarrow{\gamma^P_G} P(a) = \int_{G \in \text{Gr}_c^G(\frac{T}{\gamma})} J[G] \otimes P[G] \xrightarrow{\gamma^P_B} P[\frac{T}{\gamma}] = P[B]
\]

in which \( \gamma^P_D \) is a dioperadic composition and \( \gamma^P_B \) is a contraction \([YJ15]\) (p.216) of the wheeled properad \( P \).
4.2. Factoring the Counit

For a fixed pasting scheme $G = (S, G)$, here we observe that for a $G$-prop $P$, the counit of the free-forgetful adjunction induced by the inclusion $G_0 \leq G$ defined below naturally factors through the $W$-construction.

**Definition 4.2.1.** Suppose $G$ is a $C$-colored pasting scheme. Define the pasting scheme $G_0 \leq G$ as consisting of all $G \in G$ with $|G| = 0$, i.e., no ordinary internal edges.

**Example 4.2.2.** For the $C$-colored pasting schemes $G$ of unital linear graphs, of unital trees, of simply connected graphs, and of connected wheel-free graphs, the pasting scheme $G_0$ consists of
- exceptional edges $\uparrow_c$ for $c \in C$;
- permuted corollas with profiles allowed by the pasting scheme $G$.

**Example 4.2.3.** For the $C$-colored pasting schemes $G$ of wheeled trees and of connected wheeled graphs, the pasting scheme $G_0$ consists of
- exceptional edges $\uparrow_c$ for $c \in C$;
- exceptional loops $\sigma_c$ for $c \in C$;
- permuted corollas with profiles allowed by the pasting scheme $G$.

Recall that $S$, which is part of the pasting scheme $G$, is a replete and full sub-groupoid of $\Sigma_C^{op} \times \Sigma_C$.

**Definition 4.2.4.** A pointed $\Sigma_C$-bimodule in $M$ is a pair $(X, 1)$ in which:
1. $X \in M^S$, called a $\Sigma_C$-bimodule.
2. $1_c : 1 \longrightarrow X(\uparrow_c)$ is a map, called the $c$-colored unit, for each $c \in C$.

A map of pointed $S$-bimodules is a map of $\Sigma_C$-bimodules that preserves the colored units. The category of pointed $\Sigma_C$-bimodules in $M$ is denoted by $M^{\Sigma_C}$.

**Definition 4.2.5.** A round bimodule in $M$ is a triple $(X, 1, 0)$ consisting of
1. a pointed $\Sigma_C$-bimodule $(X, 1)$ in $M$ and
2. a map $0_c : 1 \longrightarrow X(\uparrow_c)$ for each $c \in C$.

A map of round bimodules is a map of $\Sigma_C$-bimodules that preserves the colored units and the $0_c$. The category of round bimodules in $M$ is denoted by $M^{\Sigma_C}$.

**Example 4.2.6.** If $G$ is a $C$-colored unital pasting scheme, then each $G$-prop $P$ in $M$ has an underlying pointed $\Sigma_C$-bimodule given by the $G$-prop structure maps $\gamma_P^G$ with
- $G = \uparrow_c$ (exceptional edges) for $c \in C$ for the colored units and
- $G = \sigma_C(\omega_d) \tau$ (permuted corollas) for the $\Sigma_C$-bimodule structure, for all pairs $(\omega, \uparrow_c)$ of $C$-profiles and permutations for which $\sigma_C(\omega_d) \tau$ is defined.

For example:
1. For the pasting schemes of connected wheel-free graphs (for properads), of simply connected graphs (for dioperads), of unital trees (for operads), and of unital linear graphs (for small categories), $G_0$-props are precisely pointed $\Sigma_C$-bimodules.
2. For the pasting schemes of connected wheeled graphs (for wheeled properads) and of wheeled trees (for wheeled operads), a $G_0$-prop is precisely a round bimodule with $0_c$ the $G$-prop structure map $\gamma_Q^G$ for $c \in C$. 

Remark 4.2.7. The category $M^S$ of $\Sigma_\mathcal{C}$-bimodules in $M$ is a diagram category in $M$. It is also the category of Cor$_S$-props for the minimal pasting scheme Cor$_S$ consisting of only the permuted corollas (Example 1.4.4). Adding exceptional edges $\uparrow_c$ of all colors to this minimal pasting scheme, we obtain a pasting scheme Cor$_*$ whose category of generalized props is the category of pointed $\Sigma_\mathcal{C}$-bimodules. If, furthermore, we add all the exceptional loops $Q_c$ for $c \in \mathcal{C}$, then we obtain a pasting scheme Cor$_Q$ whose category of generalized props is the category of round bimodules. Moreover, both $M^S_*$ and $M^Q$ are under-categories of $M^S$.

Definition 4.2.8. For each pair of $\mathcal{C}$-profiles $(\xi, \eta) \in S$, define the small category $E(\xi, \eta)$ as the extension category in Def. 1.7.3 for the inclusion $G_0 \leq G$ of pasting schemes.

So $E(\xi, \eta)$ has object set $G(\xi, \eta)$, and a map has the form

$$(H_v)_{v \in K} : K(H_v) \longrightarrow K$$

with each $H_v \in G_0(v)$. Composition is given by graph substitution in $G_0$, and identities are families of corollas.

The following is the special case of Lemma 1.7.4 for the inclusion $G_0 \leq G$.

Lemma 4.2.9. Suppose $G$ is a $\mathcal{C}$-colored pasting scheme. Then the forgetful functor from $G$-props to $G_0$-props in $M$ admits a left adjoint $F^G$ with entries given by

$$F^G X(\xi, \eta) = \operatorname{colim}_{K \in E(\xi, \eta)} X[K].$$

Example 4.2.10. To get a feel for the extension category $E(\xi, \eta)$ and the colimit in Lemma 4.2.9 consider the setting of Examples 3.1.3 and 3.4.6 with $G = \text{ULin}$ the one-colored pasting scheme of unital linear graphs and $\text{ULin}$ as the only substitution category. Suppose $M$ is the symmetric monoidal category $\text{Ch}(k)$, so we can talk about underlying set and elements. As discussed before, $\text{ULin}$-props in $M$ are monoids in $\text{Ch}(k)$ (i.e., differential graded algebras). The objects in the extension category are the linear graphs $L_n$ for $n \geq 0$.

The pasting scheme $G_0$ contains only the exceptional edge $L_0 = \uparrow$ and the linear graph $L_1$ with one vertex. Graph substitution involving $L_1$, which is a corolla, has no effect. So maps in the extension category $E$ are generated by the maps

$$L_{n-1} = L_n(\uparrow) \longrightarrow L_n \quad \text{for} \quad n \geq 1,$$

where $\uparrow$ is substituted into any of the $n$ vertices in the linear graph $L_n$. Here is an example with $n = 3$:

```
```

where $L_3$ and $L_2$ are as shown.
A \( G_0 \)-prop in \( \text{Ch}(k) \) is a pair \( (X, e) \) with \( X \in \text{Ch}(k) \) and \( e \in X_0 \). For each \( n \geq 0 \), we have that \( X[L_n] = X^{\otimes n} \). The monoid \( F^G X \) in Lemma 4.2.9 can be written as the quotient
\[
F^G X = \left( \bigoplus_{n \geq 0} X^{\otimes n} \right) / \sim
\]
in which the quotient is generated by the relation
\[
x_1 \otimes \cdots \otimes x_{i-1} \otimes e \otimes x_{i+1} \otimes \cdots \otimes x_n \sim x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes \cdots \otimes x_n.
\]
In particular, when \( n = 1 \) this means that \( e \in X \subseteq X^{\otimes 1} \) is identified with \( 1 \in k = X^{\otimes 0} \).

Example 4.2.11. As in Examples 3.1.8 and 3.4.10, consider the pasting scheme \( G = \text{Gr}^C \) of \( C \)-colored connected wheeled graphs, so \( \text{Gr}^C \)-props are \( C \)-colored wheeled properads in \( M \). The pasting scheme \( G_0 \) contains
\[ \text{●} \text{the exceptional edges } ↑ c \text{ for } c \in C, \\
\text{●} \text{the exceptional loops } \downarrow c \text{ for } c \in C, \text{ and} \\
\text{●} \text{the permuted corollas } \sigma C((\leq \varnothing)) \tau \text{ for all pairs of } \mathcal{C} \text{-profiles } (\varnothing) \text{ and permutations } (\sigma, \tau) \in \Sigma_{\varnothing} \times \Sigma_{\varnothing}.
\]
For a pair of \( \mathcal{C} \)-profiles \( (\varnothing) \), the objects in the extension category \( E((\varnothing)) \) are the graphs \( G \in \text{Gr}_c^C((\varnothing)) \). By associativity of graph substitution, the maps in the extension category \( E((\varnothing)) \) are generated by the maps
\[
K = G(H) \longrightarrow G \in \text{Gr}_c^C((\varnothing))
\]
where \( H \in G_0(v) \) is substituted into a chosen vertex \( v \in G \) and a corolla is substituted into all other vertices in \( G \). Let us consider this map for the three possible choices of \( H \).

The only map \([4.2.12]\) with \( H = \downarrow c \) for some \( c \in \mathcal{C} \) is the map
\[
Q_c = C((\varnothing, c))(Q_c) \longrightarrow C((\varnothing, c))
\]
in which \( C((\varnothing, c)) \) is an isolated vertex.

If \( H \) is the exceptional edge \( \uparrow c \) for some \( c \in \mathcal{C} \), then the vertex \( v \in G \) has profile \( (\cdot) \). So \( v \) has one incoming flag and one outgoing flag with the same color \( c \). The graph \( K = G(\uparrow c) \) is obtained from \( G \) by removing the vertex \( v \) and connecting the two adjacent edges. Here is one such example:

Another example arises when \( G \) is a contracted corolla \( \xi_1 C((\varnothing, c)) \) whose unique vertex has profile \( (\cdot) \) and \( K \) is the exceptional wheel \( Q_c \):
If $H$ is the permuted corolla $\sigma C(\xi)\tau$, then $v$ has profile $(\tau \sigma \xi)$. The graph

$$K = G(\sigma C(\xi)\tau)$$

is obtained from $G$ by permuting the listing at $v$ using $(\sigma^{-1}, \tau^{-1})$, so its profile in $K$ is $(\tau \sigma \xi)$.

We usually omit writing forgetful functors. So for a $G$-prop $P$, its underlying $G_0$-prop is also denoted by $P$, and $F^G P$ denotes the left adjoint $F^G$ applied to the underlying $G_0$-prop of $P$.

**Motivation 4.2.13.** Suppose $G$ is a $C$-colored pasting scheme and $P$ is a $G$-prop. Recall the free $G$-prop $F^G P$ of the underlying $G_0$-prop of $P$. By Lemma 4.2.9 we can think of $F^G P$ as the space of decorated graphs $P[G]$ with identifications parametrized by the extension categories. Given such a decorated graph, if we assign each ordinary internal edge length 1, then we obtain a sub-object of the decorated graph $(J \otimes P)[G]$ induced by the map $1 : 1 \longrightarrow J$. The $W$-construction is the space of decorated graphs $(J \otimes P)[G]$ with identifications parametrized by the substitution categories, which contain the extension categories as subcategories. So this process of assigning length 1 to all ordinary internal edges should yield a map of $G$-props from $F^G P$ to the $W$-construction of $P$. This is made precise in the following observation.

**Theorem 4.2.14.** Suppose $G$ is a $C$-colored pasting scheme, and $P$ is a $G$-prop in $M$. Then there exists a map

$$\delta : F^G P \longrightarrow W(G, J, P)$$

of $G$-props determined by the commutative diagrams

$$
\begin{array}{ccc}
P[G] & \overset{\otimes^1}{\longrightarrow} & (J \otimes P)[G] \\
\omega_G \downarrow & & \downarrow \omega_G \\
(F^G P)[(\xi)] & \overset{\delta}{\longrightarrow} & W(G, J, P)[(\xi)]
\end{array}
$$

for $G \in G[(\xi)]$, where each $\omega_G$ is the natural map and $1 : 1 \longrightarrow J$.

**Proof.** Suppose $(H_v) : G \longrightarrow K$ is a map in the extension category $E[(\xi)]$, so each $H_v \in G_0(v)$. To show that $\delta$ is well-defined, we must show that the outermost
diagram in

\[(4.2.16)\]

\[
\begin{array}{ccc}
\bigotimes_{v \in K} P[H_v] & \xrightarrow{\otimes 1_{[K]}} & (J \otimes P)[H_v] \\
\downarrow & & \downarrow \omega_{H_v} \\
\bigotimes_{v \in K} P[K(H_v)] & \xrightarrow{\otimes 1_{[K(H_v)]}} & (J \otimes P)[K(H_v)] \\
\end{array}
\]

is commutative, in which \( W = W(\mathcal{G}, J, P) \) and \( 1 : \mathbb{1} \rightarrow J \).

- The left trapezoid is commutative by definition.
- The right trapezoid is commutative by the definition of \( W(\mathcal{G}, J, P) \) as a coend over the substitution category \( \mathcal{G}(\mathcal{G}, J, P) \).
- For the top triangle, for each connected component of each \( H_v \) that is a permuted corolla, the graph substitution simply reorders the vertex listing at \( v \), and no ordinary internal edges are affected. A connected component of each \( H_v \) that is an exceptional wheel does not affect the ordinary internal edges in \( G \) and \( K \), so there is no effect with respect to \( J \). For connected components of \( H_v \) that are exceptional edges, the top triangle is commutative because 1 is an absorbing element of the segment \( J \) and \( \epsilon \circ 1 = \text{Id} \).

We have shown that the diagram \((4.2.16)\) is commutative, so the map \( \delta \) is well-defined.

To show that \( \delta \) is a map of \( \mathcal{G} \)-props, first recall from [YJ15] (Lemma 12.6) that the \( \mathcal{G} \)-prop structure on \( F^G \mathcal{P} \) is determined by the commutative diagram

\[
\begin{array}{ccc}
\bigotimes_{v \in K} P[H_v] & \xrightarrow{\otimes 1_{[K]}} & (F^G \mathcal{P})[K] = \bigotimes_{v \in K} (F^G \mathcal{P})(v) \cong \text{colim}_{\{H_v \in E(v)\}_{v \in K}} \bigotimes_{v \in K} P[H_v] \\
\downarrow & & \downarrow \gamma^K_{K(H_v)} \\
\bigotimes_{v \in K} P[K(H_v)] & \xrightarrow{\otimes 1_{[K(H_v)]}} & (F^G \mathcal{P})[K(H_v)] = \text{colim}_{G \in \mathcal{E}(\mathcal{G})} P[G] \\
\end{array}
\]

for \( K \in \mathcal{G}(\mathcal{G}, J, P) \). Since the \( \mathcal{G} \)-prop structure on \( W(G, J, P) \) is defined by the maps in \((3.5.6)\), it suffices to observe that the diagram

\[
\begin{array}{ccc}
\bigotimes_{v \in K} P[H_v] & \xrightarrow{\otimes 1_{[K(H_v)]}} & \bigotimes_{v \in K} (J \otimes P)[H_v] \\
\downarrow & & \downarrow \pi \\
\bigotimes_{v \in K} P[K(H_v)] & \xrightarrow{\otimes 1_{[K(H_v)]}} & (J \otimes P)[K(H_v)] \\
\end{array}
\]

is commutative.

As a special case of Theorem \([1.7.1]\), the inclusion

\[G_0 \subseteq \mathcal{G}\]
of pasting schemes induces a free-forgetful adjunction
\[
\text{Prop}^G(M) \xrightarrow{\delta} \text{Prop}^G(M).
\]

The next result says that the counit \(F \delta G \to \text{Id}\) of this adjunction factors through the \(W\)-construction.

**Corollary 4.2.17.** Suppose \(G\) is a \(\mathcal{C}\)-colored pasting scheme, and \(P\) is a \(G\)-prop in \(M\). Then the counit of the adjunction \(F \to G\) factors as the composite

\[
\begin{array}{ccc}
F \delta P & \to & W(G, J, P) \\
\downarrow & & \downarrow \\
\text{counit} & & \text{counit}
\end{array}
\]

with \(\eta\) the augmentation in Prop. 4.1.2.

**Proof.** The map \(\eta\) is determined in each entry by the composite \(\gamma^P \circ \epsilon^G\) for \(G \in G(\mathcal{C})\). Since \(\epsilon^1 = \text{Id}\), the composite \(\gamma^P \circ \epsilon^G\) at each entry is determined by \(\gamma^P\) for \(G \in G(\mathcal{C})\), which agrees with the counit of the adjunction. \(\Box\)

**Example 4.2.18.** Recall the setting of Example 4.2.10 with \(G = \text{ULin}\) the one-colored pasting scheme of unital linear graphs, \(\text{ULin}\) as the only substitution category, and \(M\) the symmetric monoidal category \(\text{Ch}(k)\). For a monoid \(P\) in \(M\), the monoid \(F \delta G P\) in Lemma 4.2.9 is a quotient
\[
F \delta G P = \left( \bigoplus_{n \geq 0} P^\otimes n \right) / \sim
\]
in which the quotient is generated by the unit element \(e \in P_0\).

As we pointed out in Example 3.4.6, the \(W\)-construction \(3.4.3\) is the coend
\[
WP = \int^{L_n \in \text{ULin}} J[L_n] \otimes P[L_n].
\]

The map \(\delta : F \delta G P \to WP\) is determined by the commutative diagrams

\[
\begin{array}{ccc}
P[L_n] & \xrightarrow{1^\otimes[L_n]} & J^\otimes[L_n] \otimes P^\otimes n \\
\downarrow \omega_L & & \downarrow \omega_L \\
(F \delta G P) & \xrightarrow{\delta} & WP
\end{array}
\]
for \(n \geq 0\). The top horizontal map is induced by
\[
1^\otimes[L_n] = \begin{cases} 
\text{Id} & \text{if } n = 0; \\
1^\otimes[L_n-1] : 1 \cong 1^\otimes[L_n-1] & \text{if } n \geq 2.
\end{cases}
\]

**Example 4.2.19.** Suppose \(G\) is the \(\mathcal{C}\)-colored pasting scheme of connected wheeled graphs, and \(P\) is a \(\mathcal{C}\)-colored wheeled properad in \(M\). Suppose
\[
P[G] = P(r) \otimes P(s) \otimes P(t)
\]
is the decorated graph on the left:
4.3. Naturality

The $W$-construction is natural in all three variables in the following sense.

**Motivation 4.3.1.** We have defined the $W$-construction of a $\mathcal{G}$-prop $P$ entrywise as a coend

$$W(\mathcal{G}, J, P) = \int_{\mathcal{G}}^{\mathcal{G}_2} J[\mathcal{G}] \otimes P[\mathcal{G}].$$

By the various naturality properties of coends, it should come as no surprise that the $W$-construction is natural with respect to the various ingredients. We spell out these naturality properties in the next observation.

**Corollary 4.3.2.** Suppose $\mathcal{G} \leq \mathcal{G}'$ is a pair of $\mathcal{C}$-colored pasting schemes, and $J$ is a commutative segment in $\mathcal{M}$.

1. If $P$ is a $\mathcal{G}'$-prop in $\mathcal{M}$, then there is a natural induced map of $\mathcal{G}$-props

$$W(\mathcal{G}, J, P) \longrightarrow W(\mathcal{G}', J, P)$$

in which $U$ is the forgetful functor from $\mathcal{G}'$-props to $\mathcal{G}$-props (1.7.2).

2. If $P$ is a $\mathcal{G}$-prop in $\mathcal{M}$, then each map $j : J \longrightarrow J'$ of commutative segments induces a natural map of $\mathcal{G}'$-props

$$W(\mathcal{G}, J, P) \longrightarrow W(\mathcal{G}, J', P).$$

3. If $P \longrightarrow Q$ is a map of $\mathcal{G}$-props, then there is a natural induced map of $\mathcal{G}'$-props

$$W(\mathcal{G}, J, P) \longrightarrow W(\mathcal{G}, J, Q).$$

**Proof.** For (1) observe that the entries and the structure maps of the $\mathcal{G}$-prop $\mathcal{U}P$ are all from $\mathcal{P}$ and that each substitution category $\mathcal{G}_2^{(2)}$ is a sub-category of $\mathcal{G}'_2^{(2)}$. So the required map is induced entrywise by the universal property of the coend $W(\mathcal{G}, J, \mathcal{U}P)^{(2)}$. 

Then the commutative diagram (4.2.15) takes the form

$$\begin{array}{c}
P[\mathcal{G}] \cong \mathbf{1}^{\otimes \mathcal{G}} \otimes P(\mathcal{G}) \otimes P(\mathcal{G}) \otimes P(\mathcal{G}) \\
\xrightarrow{\mathbf{1}^\otimes \mathcal{G}} \mathbf{1}^{\otimes \mathcal{G}} \otimes P[\mathcal{G}] \\
\xrightarrow{\mathbf{1}^\otimes \mathcal{G}} (J \otimes P)[\mathcal{G}]
\end{array}$$

The top horizontal map is illustrated as the top composite in the previous picture.
For (2) simply note that a map \( j : J \rightarrow J' \) of commutative segments induces a natural transformation
\[
J \Rightarrow J' : G(\xi_2)^{\text{op}} \rightarrow M
\]
for each pair of \( \mathcal{C} \)-profiles \( (\xi) \).

For (3) observe that a map of \( \mathcal{G} \)-props \( P \rightarrow Q \) induces a natural transformation
\[
P \Rightarrow Q : G(\xi_2) \rightarrow M
\]
for each pair of \( \mathcal{C} \)-profiles \( (\xi) \). \( \square \)

**Example 4.3.3.** Consider the setting of Corollary 4.3.2 (1) with \( \mathcal{G} \leq \mathcal{G}' \) a pair of \( \mathcal{C} \)-colored pasting schemes. Suppose \( P \) is a \( \mathcal{G}' \)-prop in \( \mathcal{M} \) and \( U \) is the forgetful functor from \( \mathcal{G}' \)-props to \( \mathcal{G} \)-props. For each pair of \( \mathcal{C} \)-profiles \( (\xi) \) for the pasting scheme \( \mathcal{G} \), the \( (\xi) \)-entry of the map in Corollary 4.3.2 (1) is the uniquely induced bottom horizontal map in the commutative diagram
\[
\begin{array}{ccc}
(J \otimes UP)[G] & \xrightarrow{\sim} & (J \otimes P)[G] \\
\omega_G \downarrow & & \downarrow \omega_G \\
W(\mathcal{G}, J, UP)(\xi) & \rightarrow & UW(\mathcal{G}', J, P)(\xi)
\end{array}
\]
that holds for each \( G \in G(\xi) \). For instance, suppose \( \mathcal{G} = \text{Gr}_c^+ \) and \( \mathcal{G}' = \text{Gr}_c^c \) are the \( \mathcal{C} \)-colored pasting schemes of connected wheel-free graphs and of connected wheeled graphs, respectively. Then \( W(\mathcal{G}, J, UP) \) is a properad, and \( UW(\mathcal{G}', J, P) \) is the properad obtained from the wheeled properad \( W(\mathcal{G}', J, P) \) by applying the forgetful functor.

That the maps \( \iota \) form a map of \( \mathcal{G} \)-props is a consequence of the commutative diagram
\[
\begin{array}{ccc}
\otimes \left( J \otimes UP \right)[H_v] & \xrightarrow{\sim} & \otimes \left( J \otimes P \right)[H_v] \\
\pi \downarrow & & \downarrow \pi \\
(J \otimes UP)[G(H_v)] & \xrightarrow{\sim} & (J \otimes P)[G(H_v)]
\end{array}
\]
for each choice of \( H_v \in \mathcal{G}(v) \) for \( v \in G \), where \( \pi \) is the map in (3.5.6).

**Example 4.3.4.** Consider the setting of Corollary 4.3.2 (2) with \( P \) a \( \mathcal{G} \)-prop and \( J \) a commutative segment in \( \mathcal{M} \). Consider the commutative segment
\[
\begin{array}{ccc}
1 \sqcup 1 & \xrightarrow{=} & 1 \sqcup 1 \\
\text{fold} & & \rightarrow 1
\end{array}
\]
with multiplication
\[
(1_0 \sqcup 1_1)^{\otimes 2} \cong (1_0 \otimes 1_0) \cup (1_0 \otimes 1_1) \cup (1_1 \otimes 1_0) \cup (1_1 \otimes 1_1) \rightarrow 1_0 \sqcup 1_1
\]
sending \( 1_0 \otimes 1_0 \) \( \cong \) \( 1 \sqcup 1 \) to \( 1_0 \) and the other three coproduct summands to \( 1_1 \). For each graph \( G \in \mathcal{G}(\xi_2) \), we have that
\[
(1_0 \sqcup 1_1)[G] = (1_0 \sqcup 1_1)^{\otimes |G|} \cong \coprod_{v \in |G|} \bigoplus_{\{0,1\}} 1_{\nu(v)}.
\]

There is a map of commutative segments
\[
(0,1) : 1 \sqcup 1 \rightarrow J.
4.4. Changing the Base Categories

For each pair of $\mathcal{C}$-profiles $(\mathcal{P})$, the $(\mathcal{P})$-entry of the map in Corollary 4.3.2(2) is the uniquely induced bottom horizontal map in the commutative diagram

\[
\begin{array}{c}
(\mathcal{P}) \cong \mathcal{P} \otimes (\mathcal{P}) \supseteq \mathcal{P} \otimes (\mathcal{P}) \supseteq \mathcal{P} \otimes (\mathcal{P}) \\
\downarrow \omega_G \downarrow \omega_G \\
W(\mathcal{G}, \mathcal{P}) \otimes (\mathcal{P}) \rightarrow W(\mathcal{G}, \mathcal{P}) \otimes (\mathcal{P})
\end{array}
\]

that holds for each $G \in \mathcal{G}(\mathcal{P})$, where in the top horizontal map $\nu(e) : I \rightarrow J$.

Example 4.3.5. Consider the setting of Corollary 4.3.2(3) with $f : \mathcal{P} \rightarrow \mathcal{Q}$ a map of $\mathcal{G}$-props. For each pair of $\mathcal{C}$-profiles $(\mathcal{P})$, the $(\mathcal{P})$-entry of the map in Corollary 4.3.2(3) is the uniquely induced bottom horizontal map in the commutative diagram

\[
\begin{array}{c}
(J \otimes \mathcal{P})(G) \xrightarrow{\otimes_{e \in G} f} (J \otimes \mathcal{Q})(G) \\
\downarrow \omega_G \downarrow \omega_G \\
W(\mathcal{G}, J \otimes \mathcal{P})(\mathcal{P}) \xrightarrow{f_\ast} W(\mathcal{G}, J \otimes \mathcal{Q})(\mathcal{P})
\end{array}
\]

that holds for each $G \in \mathcal{G}(\mathcal{P})$. That the maps $f_\ast$ form a map of $\mathcal{G}$-props is a consequence of the commutative diagram

\[
\begin{array}{c}
\otimes_{e \in G} (J \otimes \mathcal{P})(H_v) \xrightarrow{\otimes_{e \in G} f} \otimes_{e \in G} (J \otimes \mathcal{Q})(H_v) \\
\downarrow \pi \downarrow \pi \\
(J \otimes \mathcal{P})(G(H_v)) \xrightarrow{\otimes_{e \in G} f} (J \otimes \mathcal{Q})(G(H_v))
\end{array}
\]

for each choice of $H_v \in \mathcal{G}(v)$ for $v \in G$.

Example 4.3.6. Both maps in Corollary 4.1.4 are natural with respect to the generalized prop and the commutative segment. In other words, suppose $J' \rightarrow J$ is a map of commutative segments, and $f : \mathcal{P} \rightarrow \mathcal{Q}$ is a map of $\mathcal{G}$-props. Then the diagram

\[
\begin{array}{c}
\mathcal{P}(\mathcal{P}) = (J' \otimes \mathcal{P})(C) \xrightarrow{\omega_C} W(\mathcal{G}, J', \mathcal{P})(\mathcal{P}) \rightarrow \mathcal{P}(\mathcal{P}) \\
\mathcal{P}(\mathcal{P}) = (J \otimes \mathcal{P})(C) \xrightarrow{\omega_C} W(\mathcal{G}, J, \mathcal{P})(\mathcal{P}) \rightarrow \mathcal{P}(\mathcal{P}) \\
\mathcal{Q}(\mathcal{P}) = (J \otimes \mathcal{Q})(C) \xrightarrow{\omega_C} W(\mathcal{G}, J, \mathcal{Q})(\mathcal{P}) \rightarrow \mathcal{Q}(\mathcal{P})
\end{array}
\]

is commutative, where the middle vertical maps are those in Corollary 4.3.2.

4.4. Changing the Base Categories

Here we observe that the $W$-construction is compatible with change of base categories. Suppose both $\mathcal{M}$ and $\mathcal{N}$ are cocomplete symmetric monoidal categories in which the monoidal product commutes with colimits in both variables.
4. CATEGORICAL PROPERTIES OF THE BOARDMAN-VOGT CONSTRUCTION

**Notation 4.4.1.** For a lax monoidal functor $f: M \to N$ [Mac98] (XI.2), its structure map

$$fX \otimes fY \to f(X \otimes Y)$$

for $X, Y \in M$ and its iterates will be denoted by $f_1$.

**Motivation 4.4.2.** Suppose $P$ is a $G$-prop in $M$ for some pasting scheme $G$, and $f: M \to N$ is a symmetric monoidal functor. There are two ways to make a $G$-prop in $N$ using $P$ and the $W$-construction. First, we may send $P$ to the $G$-prop $fP$ in $N$, and then take its $W$-construction in $N$. Alternatively, we may send the $W$-construction of $P$ to $N$ using $f$. Since the $W$-construction and its structure maps are all defined naturally, the two $G$-props in $N$ obtained previously should be closely related. This is made precise in the following observation.

**Theorem 4.4.3.** Suppose $f: M \to N$ is a unit-preserving lax symmetric monoidal functor, $J$ is a commutative segment in $M$, and $P$ is a $G$-prop in $M$ with $G$ a $C$-colored pasting scheme. Then there is a naturally induced commutative diagram

\[
\begin{array}{ccc}
W(G, fJ, fP) & \xrightarrow{f_*} & fW(G, J, P) \\
\eta^P \downarrow & & \downarrow f\eta^P \\
fP & \xrightarrow{\cong} & fP
\end{array}
\]

of $G$-props in $N$ such that $f_*$ is induced by $f$ and that $\eta^P$ and $f\eta^P$ are the augmentations from Prop. 4.1.2.

**Proof.** The assumption on $f$ ensures that $fJ$ is a commutative segment in $N$ and that $f$ sends a $G$-prop in $M$ to a $G$-prop in $N$ [YJ15] (Theorem 12.11(1)). For $G \in \mathcal{G}(\mathcal{G})$, the corresponding $\mathcal{G}$-prop structure map of $fP$ is the composite

\[
(fP)[G] \xrightarrow{\gamma^P_G} fP(\mathcal{G}) \xrightarrow{f\gamma^P_G} f(P[G])
\]

in which $\gamma^P_G$ is the $G$-prop structure map of $P$ corresponding to $G$. This also applies to the $\mathcal{G}$-prop $fW(G, J, P)$ in $N$ (Theorem 3.5.17).

First we define the top horizontal map $f_*$ in (4.4.4). For $G \in \mathcal{G}(\mathcal{G})$ we define the restriction of $f_*$ by insisting that the diagram

\[
\begin{array}{ccc}
W(G, fJ, fP)(\mathcal{G}) & \xrightarrow{f_*} & fW(G, J, P)(\mathcal{G}) \\
\omega_G \downarrow & & \downarrow f\omega_G \\
(fJ)[G] \otimes (fP)[G] & \xrightarrow{f_1} & f((J \otimes P)[G])
\end{array}
\]
be commutative. Given any map \((H_v)_v : G \rightarrow K \in \underline{G}(\sqcup)\), the commutativity of the diagram

\[
\begin{array}{c}
(fJ)[K] \otimes (fP)[G] \xrightarrow{f_1} (fJ)[K] \otimes f(P[H_v]) \xrightarrow{\otimes f_{\gamma_H_v}} (fJ)[K] \otimes (fP)[K] \\
\downarrow f(J) \quad \quad \downarrow f(J) \\
(fJ)[G] \otimes (fP)[G] \xrightarrow{f_1} f((J \otimes P)[G]) \xrightarrow{f_{\gamma_H_v}} fW(G, J, P)(\sqcup)
\end{array}
\]

shows that the map \(f_*\) is well-defined entrywise. Here the left square and the top right rectangle are commutative by the naturality of \(f\). The lower right rectangle is commutative by the definition of \(W(G, J, P)(\sqcup)\) as a coend over \(\underline{G}(\sqcup)\).

For a map \((H_v)_v : G \rightarrow K \in \underline{G}(\sqcup)\), the commutativity of the diagram

\[
\begin{array}{c}
\otimes \left( fJ \otimes fP \right)[H_v] \xrightarrow{\otimes f_1} \otimes f((J \otimes P)[H_v]) \xrightarrow{\otimes f_{\gamma_H_v}} (fW^M)[K] \\
\downarrow f(J) \quad \quad \downarrow f(J) \\
(fJ \otimes fP)[G] \xrightarrow{f_1} f\left( (J \otimes P)[G] \right) \xrightarrow{f_{\gamma_H_v}} fW^M[G]
\end{array}
\]

shows that \(f_*\) is a map of \(G\)-props in \(\mathbb{N}\), in which

\[
W^M = W(G, J, P) \quad \text{and} \quad W^N = W(G, fJ, fP).
\]

Here the left (resp., right) wedge is commutative by the definition of \(\gamma^W(3.5.6)\) (resp., \(\gamma^{fW^M}(4.4.5)\)). The sub-diagram 1 (resp., 2) is commutative by the definition of \(f_* (4.4.6)\) (resp., \(\gamma^W(3.5.6)\)). The other three sub-diagrams are commutative by the naturality of \(f\).
Finally, the commutativity of the diagram (4.4.4) is a consequence of the commutativity of the diagram

\[
\begin{array}{ccc}
(fJ \otimes fP)[G] & \xrightarrow{f_1} & f\left((J \otimes P)[G]\right) \\
\downarrow^{(fJ) \otimes (fP)} & & \downarrow^{f_{\omega G}} \\
(fP)[G] & \xrightarrow{f_1} & f(P[G]) \\
\end{array}
\]

for any \( G \in \mathcal{G}(\mathbb{Z}) \). Here the left square is commutative by the naturality of \( f \), and the right square is commutative by the definition of \( \eta \) (4.1.3).  \( \square \)

**Corollary 4.4.7.** In Theorem 4.4.3, if \( f \) is a strong symmetric monoidal colimit-preserving functor, then the map

\[ f_* : W(\mathcal{G}, fJ, fP) \longrightarrow fW(\mathcal{G}, J, P) \]

is an isomorphism of \( \mathcal{G} \)-faits in \( \mathbb{N} \).

**Proof.** Since \( f \) is strong symmetric monoidal, the monoidal structure map \( f_1 \) in the definition of the map \( f_* \) (4.4.6) is an isomorphism. Furthermore, since \( f \) is colimit-preserving, it preserves coends, so \( f_* \) is entrywise an isomorphism.  \( \square \)

**Example 4.4.8.** Here are some examples of strong symmetric monoidal, unit-preserving, and colimit-preserving functors, for which Corollary 4.4.7 applies. Note that left adjoints preserve colimits.

1. Suppose \( f : R \longrightarrow S \) is a map of unital commutative rings with \( \text{Mod}(R) \) and \( \text{Mod}(S) \) the categories of \( R \)-modules and of \( S \)-modules, respectively. There is an extension-restriction of scalars adjunction

\[
\text{Mod}(R) \xrightarrow{S \otimes_R -} \text{Mod}(S)
\]

in which the left adjoint \( S \otimes_R - \) is a strong symmetric monoidal, unit-preserving, and colimit-preserving functor.

2. The geometric realization functor from simplicial sets to compactly generated Hausdorff spaces is a strong symmetric monoidal, unit-preserving, and colimit-preserving functor [GZ67].

3. Suppose \( M \) is a symmetric monoidal category. Then the functor

\[ X \longrightarrow \coprod_{x \in X} 1 \]

from the Cartesian category of sets to \( M \) is a strong symmetric monoidal, unit-preserving, and colimit-preserving functor.

A homotopical version of Theorem 4.4.3 will be given in Theorem 7.3.5 below.
CHAPTER 5

Filtering the Boardman-Vogt Construction

As before \((M, \otimes, 1)\) is a cocomplete symmetric monoidal category in which the monoidal product commutes with colimits in both variables, and \((J, \mu, 0, 1, \epsilon)\) is a commutative segment in \(M\). Suppose \(G = (S, G)\) is a pasting scheme. Recall that each entry of our \(W\)-construction \(W(G, J, P)\) is defined as a coend over a substitution category.

In Section 5.1 we construct a natural filtration of the \(W\)-construction using coends over smaller substitution categories. In Section 5.2 we define the objects that will allow us to understand the filtration strata as pushouts. In Section 5.3 and Section 5.4 we show that each pair of consecutive strata of the filtration fits into a specific pushout. In Section 5.5 we factor each map in the filtration into two maps. Later we will use this refined filtration to establish nice homotopical properties of the \(W\)-construction. In Section 5.6 we explain the underlying categorical setting needed for this entire machinery to work.

5.1. Strata of the Construction

For a graph \(G\) recall that \(|G|\) denotes the set of ordinary internal edges in \(G\).

**Notation 5.1.1.** To simplify the notation we will also use \(|G|\) to denote its cardinality, i.e., the number of ordinary internal edges in \(G\).

**Motivation 5.1.2.** For a pasting scheme \(G\), the \(W\)-construction of a \(G\)-prop \(P\) in \(M\) with a commutative segment \(J\) was defined entrywise as a coend

\[
W(G, J, P) = \int_{G \in G} J[G] \otimes P[G].
\]

Eventually we will show that the \(W\)-construction is a cofibrant resolution of \(P\), provided \(P\) is mildly nice to begin with. To prove that the \(W\)-construction is a cofibrant \(G\)-prop, we will define a filtration on it in such a way that each positive filtration strata can be constructed from the previous one via a particularly nice pushout. To define this filtration on the \(W\)-construction, we filter each substitution category \(G_n(\mathcal{G})\) using graphs with at most a certain number of ordinary internal edges.

**Definition 5.1.3.** Suppose \(n\) is a non-negative integer and \(P\) is a \(G\)-prop in \(M\).

1. Define \(G_n(\mathcal{G})\) as the full sub-category of the substitution category \(G(\mathcal{G})\) (Def. 3.1.2) with objects \(G\) such that \(|G| \leq n\).
2. Define the coend

\[
W(G, J, P)_{\mathcal{G}_n(\mathcal{G})} = \int_{G \in G_n} J[G] \otimes P[G].
\]

in which on the right-hand side:

- \(P:\mathcal{G}_n(\mathcal{G}) \to M\) is the functor in Def. 3.1.2 restricted to \(G_n(\mathcal{G})\);
• $J : \mathcal{G}(\mathcal{C})^{op} \to \mathcal{M}$ is the functor in Def. 3.2.4 restricted to $\mathcal{G}(\mathcal{C})^{op}$.

3. Equip $\tilde{W}(\mathcal{G}, J, P)_n$ with the structure of a $\Sigma$-bimodule in $\mathcal{M}$ by declaring that the diagram

\[
\begin{array}{ccc}
(J \otimes P)[G] & \xrightarrow{\omega_G} & W(\mathcal{G}, J, P)_n(\mathcal{C}) \\
\downarrow & & \downarrow_{(\tau, \sigma)} \\
(J \otimes P)[\sigma G \tau] & \xrightarrow{\omega_{\sigma G \tau}} & W(\mathcal{G}, J, P)_n(\mathcal{C})
\end{array}
\]

be commutative for each object $G \in \mathcal{G}(\mathcal{C})$, in which each map $\omega$ is the natural map to a coend. The identity map on the left makes sense because changing the graph profiles of $G$ does not change its vertex profiles and set of ordinary internal edges.

**Example 5.1.6.** Suppose $\mathcal{G}$ is the $\mathcal{C}$-colored pasting scheme $\text{ULin}$ of unital linear graphs, so $\mathcal{G}$-props in $\mathcal{M}$ are small categories enriched in $\mathcal{M}$ with object set $\mathcal{C}$. For $c, d \in \mathcal{C}$ and $n \geq 0$, the category $\text{ULin}_n(\mathcal{C})$ contains all the linear graphs $L_m$ with $m \leq n + 1$ vertices whose input has color $c$ and whose output has color $d$.

\[
L_m = \begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & c_1 & c_2 & \ldots & c_{n-1} & d
\end{array}
\]

In particular, if $c = d$ then $\text{ULin}_n(\mathcal{C})$ contains the exceptional edge $\uparrow_c$. If $c \neq d$ then $\text{ULin}_n(\mathcal{C})$ does not contain any exceptional edges.

**Example 5.1.7.** Suppose $\mathcal{G}$ is the $\mathcal{C}$-colored pasting scheme $\text{Gr}_c^\uparrow$ of connected wheel-free graphs, so $\mathcal{G}$-props in $\mathcal{M}$ are $\mathcal{C}$-colored properads. The exceptional edges $\uparrow_c$ for $c \in \mathcal{C}$ are the only connected wheel-free graphs with no vertices, and therefore also no ordinary internal edges. Connected wheel-free graphs with one vertex, and therefore no ordinary internal edges, are the permuted corollas. Connected wheel-free graphs with one internal edge must have the form

\[
(5.1.8)
\]

with two vertices.

Connected wheel-free graphs with two ordinary internal edges must have one of the following four forms:

\[
(5.1.9)
\]

More information about these connected wheel-free graphs can be found in [YJ15] Section 6.4.1.

**Example 5.1.10.** Suppose $\mathcal{G}$ is the $\mathcal{C}$-colored pasting scheme $\text{Gr}_c^\uparrow$ of connected wheeled graphs, so $\mathcal{G}$-props in $\mathcal{M}$ are $\mathcal{C}$-colored wheeled properads. The exceptional edges $\uparrow_c$ and the exceptional loops $Q_c$ for $c \in \mathcal{C}$ are the only connected wheeled
5.1. STRATA OF THE CONSTRUCTION

graphs with no vertices and no ordinary internal edges. Permutated corollas are the only connected wheeled graphs with one vertex and no ordinary internal edges.

A connected wheeled graph with one ordinary internal edge either has the form (5.1.8) or is a contracted corolla:

Connected wheeled graphs with two ordinary internal edges are the ones in (5.1.9) and the following graphs:

More information about these connected wheeled graphs can be found in [YJ15] Section 6.4.2.

Motivation 5.1.11. For a \( G \)-prop \( P \) in \( M \) and a fixed \( n \), one should not expect the object \( W(\mathcal{G},J,P)_n \) to inherit a \( G \)-prop structure from the \( W \)-construction \( W(\mathcal{G},J,P) \). Indeed, the \( G \)-prop structure map in the \( W \)-construction (3.5.6) involves graph substitution. Graphs in the subcategory \( \mathcal{G}_n(\frac{2}{2}) \) of the substitution category \( \mathcal{G}(\frac{2}{2}) \) are not closed under graph substitution because graph substitution sometimes increases the number of ordinary internal edges. For example, substituting the linear graph \( L_2 \in \text{ULin}_1 \), which has one internal edge, into either vertex of another copy of \( L_2 \) yields \( L_3 \) with two internal edges.

On the other hand, changing the input/output profiles of a graph does not affect ordinary internal edges. So one can expect that the object \( W(\mathcal{G},J,P)_n \) be a \( \Sigma\mathcal{E} \)-bimodule.

Lemma 5.1.12. For each \( G \)-prop \( P \) in \( M \), \( W(\mathcal{G},J,P)_n \) is a \( \Sigma\mathcal{E} \)-bimodule in \( M \).

Proof. We need to show that the equivariant structure maps in \( W(\mathcal{G},J,P)_n \) are actually well-defined. Suppose \( (H_v): G(H_v) \longrightarrow G \) is a map in \( \mathcal{G}_n(\frac{2}{2}) \). Writing

\[
\sigma G \tau = (\sigma C \tau)(G),
\]

where \( C \) is the corolla with profiles \( (\frac{2}{2}) \), there is a map

\[
(H_v): \sigma G(H_v) \tau = (\sigma C \tau)(G)(H_v) \longrightarrow \sigma G \tau
\]
in $G_n(\mathcal{C}, \mathcal{D})$. It is enough to show that in

$$
\begin{array}{c}
J[G] \otimes P[G(H_v)] \\
\downarrow J \\
J[\sigma G\tau] \otimes P[\sigma G(H_v)\tau] \\
\downarrow J \\
(J \otimes P)[G(H_v)]
\end{array}
\xrightarrow{\otimes \gamma^p_{H_v}}
\begin{array}{c}
(J \otimes P)[\sigma G(H_v)\tau] \\
\downarrow \omega_{\sigma G(H_v)\tau} \\
W(\mathcal{G}, J, P)_n(\mathcal{C}, \mathcal{D})
\end{array}
$$

the outermost diagram is commutative. The left and the top trapezoids are commutative by inspection. The lower-right square is commutative by the coend definition of $W(\mathcal{G}, J, P)_n(\mathcal{C}, \mathcal{D})$.

\[\square\]

**Lemma 5.1.13.** Suppose $\mathcal{G}$ is a $\mathcal{C}$-colored pasting scheme. For each $\mathcal{G}$-prop $P$ in $\mathcal{M}$, the maps

$$P(\mathcal{C}) \cong (J \otimes P)[C_{\mathcal{E}(\mathcal{D})}] \xrightarrow{\omega_{\mathcal{C}(\mathcal{E}(\mathcal{D}))}} W(\mathcal{G}, J, P)_0(\mathcal{C})$$

define an isomorphism of $\Sigma_{\mathcal{E}}$-bimodules.

**Proof.** Since there are no ordinary internal edges for any $G \in G_0(\mathcal{C}, \mathcal{D})$, the functor $J$ is trivial (i.e., $J[G] = \mathbb{1}$ for all $G \in G_0(\mathcal{C}, \mathcal{D})$). By the coend definition of $W(\mathcal{G}, J, P)_0$, a map out of $W(\mathcal{G}, J, P)_0(\mathcal{C})$ is uniquely determined by a map out of $P(\mathcal{C})$ via $\omega_{\mathcal{C}(\mathcal{E}(\mathcal{D}))}$, in the sense that each map out of $(J \otimes P)[G] = P[G]$ is $\gamma^p_G$ followed by the map out of $P(\mathcal{C}) = P[C_{\mathcal{E}(\mathcal{D})}]$. So $P$ and $W(\mathcal{G}, J, P)_0$ are isomorphic entrywise.

To see that they are isomorphic as $\Sigma_{\mathcal{E}}$-bimodules, simply observe that the diagram

$$
\begin{array}{c}
P(\mathcal{C}) = P[C_{\mathcal{E}(\mathcal{D})}] \\
\downarrow \\
P[\sigma C_{\mathcal{E}(\mathcal{D})}\tau] \\
\downarrow \gamma^p_{\mathcal{C}(\mathcal{E}(\mathcal{D}))}\tau \\
P(\mathcal{C}) = P[C_{\mathcal{E}(\mathcal{D})}] \\
\downarrow \\
W(\mathcal{G}, J, P)_0(\mathcal{C})
\end{array}
\xrightarrow{\omega_{\mathcal{C}(\mathcal{E}(\mathcal{D}))}\tau} W(\mathcal{G}, J, P)_0(\mathcal{C})
$$

is commutative. Indeed, the top square is commutative by definition (5.1.5). The bottom square is proved to be commutative in the previous paragraph. \[\square\]

We now filtered the $W$-construction using the above intermediate objects.

**Motivation 5.1.14.** The $W$-construction of a $\mathcal{G}$-prop $P$ in $\mathcal{M}$ is entrywise a coend

$$W(\mathcal{G}, J, P)(\mathcal{C}) = \int^{G \in G(\mathcal{C})} J[G] \otimes P[G].$$

Each map $(H_v) : K = G(H_v) \rightarrow G$ in the substitution category $G(\mathcal{C})$ belongs to some sub-category $G_n(\mathcal{C})$ (e.g., $n = \max\{||K|,|G||\}$). So it makes sense that the $W$-construction is at least entrywise the sequential colimit of the objects $W(\mathcal{G}, J, P)_n$. 


Proposition 5.1.15. For each \(G\)-prop \(P\) in \(M\), there are a natural filtration and an isomorphism
\[
W(G, J, P)_0 \longrightarrow W(G, J, P)_1 \longrightarrow \cdots \longrightarrow \colim_n W(G, J, P)_n \cong W(G, J, P)
\]
of \(\Sigma_{\mathcal{C}}\)-bimodules in \(M\).

Proof. The filtration and the isomorphism are induced by the exhaustive filtration
\[
\mathcal{G}_0(\xi) \subseteq \mathcal{G}_1(\xi) \subseteq \mathcal{G}_2(\xi) \subseteq \cdots \subseteq \bigcup_{n=0}^{\infty} \mathcal{G}_n(\xi) = \mathcal{G}(\xi)
\]
of categories. It follows from the definition that each map
\[
W(G, J, P)_n \longrightarrow W(G, J, P)_{n+1}
\]
respects the equivariant structure. It follows from Def. 3.5.3 that the equivariant structure on \(W(G, J, P)\) is defined by the commutative diagrams
\[
\begin{array}{ccc}
(J \otimes P)[G] & \longrightarrow & \colim_{\mathcal{G}(\xi)} (J \otimes P)[G] \\
\pi = \text{Id} & \downarrow & \gamma_{\sigma C \tau} \\
(J \otimes P)[\sigma G \tau] & \longrightarrow & W(G, J, P)_{n(\xi)}
\end{array}
\]
Therefore, each map
\[
W(G, J, P)_n \longrightarrow W(G, J, P)
\]
also respects the equivariant structure.

\[
\Box
\]

5.2. Shrinking Tunnels and Ordinary Internal Edges

We now attempt to build each object \(W(G, J, P)_{n+1}\) from the previous one, for which we will need the following devices.

Motivation 5.2.1. Suppose \(P\) is a \(G\)-prop in \(M\) for some pasting scheme \(G\). In order to relate two consecutive filtration strata \(W(G, J, P)_n\) and \(W(G, J, P)_{n+1}\), we need to think about how a graph \(G\) in \(\mathcal{G}_{n+1}(\xi)\) is related to the sub-category \(\mathcal{G}_n(\xi)\). If \(G\) has at most \(n\) ordinary internal edges, then it is already in \(\mathcal{G}_n(\xi)\).

Suppose \(G\) has exactly \(n + 1\) ordinary internal edges, so it belongs to \(\mathcal{G}_{n+1}(\xi)\) but not to \(\mathcal{G}_n(\xi)\). We can consider a map \(K \longrightarrow G\) or a map \(G \longrightarrow K\) in the substitution category \(\mathcal{G}(\xi)\) where \(K\) has at most \(n\) internal edges. We first consider a map of the form \(K \longrightarrow G\). Suppose \(v\) is a vertex in \(G\) with one incoming flag and one outgoing flag with the same color \(c\) such that at least one of its incoming flag and outgoing flag is an internal edge in \(G\). Then the graph substitution \(G(\uparrow_c)\), in which the exceptional edge \(\uparrow_c\) is substituted into \(v\) and a corolla is substituted into every other vertex in \(G\), has one fewer internal edge than \(G\). In other words, we have \(G(\uparrow_c) \in \mathcal{G}_{n+1}(\xi)\), and
\[
(\uparrow_c) : G(\uparrow_c) \longrightarrow G
\]
is a map in \(\mathcal{G}(\xi)\).
For example, \(G\) below has four internal edges, while \(G(\uparrow_c)\) has three.
The exceptional edge $\uparrow_c$ corresponds to the $c$-colored unit of the $\mathcal{G}$-prop $P$. The lengths of the two internal edges in $G$ will be multiplied in $G(\uparrow_c)$. Graph substitution involving only exceptional edges will appear in the next definition. This is the first way in which a graph with $n + 1$ ordinary internal edges is related to the sub-category $\mathcal{G}_n(\hat{e})$. We will discuss a second way after the following definition.

The next definition is only needed if the pasting scheme contains all the exceptional edges.

**Definition 5.2.2.** Suppose $\mathcal{G}$ is a $\mathcal{C}$-colored unital pasting scheme, $G \in \mathcal{G}(\hat{e})$, and $P$ is a $\mathcal{G}$-prop in $M$.

1. A *tunnel* in $G$ is a vertex $v$ with precisely one incoming flag and one outgoing flag that have the same color. The set of all the tunnels in $G$ is denoted by $\text{Tun}(G)$.

2. For each non-empty set of tunnels $T$ in $G$, define
   
   $$G/T = G(\uparrow_t)_{t \in T}$$
   
   in which $\uparrow_t$ is the exceptional edge whose color is the same as that of the flags adjacent to the tunnel $t$. Since there is a map
   
   $$(\uparrow_t)_{t \in T} : G/T \longrightarrow G$$
   
   in $\mathcal{G}_n(\hat{e})$, where $|G| = n$, there is a map
   
   $$P[G/T] \cong P[G/T] \otimes \bigotimes_{t \in T} 1_{\mathcal{G}_n(\hat{e})} \longrightarrow P[G]$$
   
   given by the colored units of $P$.

3. These maps assemble to yield a map
   
   $$(5.2.3) \quad P^-[G] := \colim_{G/T \in \text{Tun}(G)} P[G/T] \longrightarrow P[G]$$
   
   in which the colimit is indexed by the category of non-empty subsets of tunnels $T$ in $G$, where a map $T \longrightarrow T'$ is a set inclusion $T' \subseteq T$.

**Convention 5.2.4.** Unless otherwise specified, we assume $\mathcal{G}$ is a $\mathcal{C}$-colored unital pasting scheme. All the constructions that follow also work if $\mathcal{G}$ is non-unital, where all the parts that involve the exceptional edges and $P^-$ are ignored.
5.2. SHRINKING TUNNELS AND ORDINARY INTERNAL EDGES

Example 5.2.5. The above definitions are most easily visualized using linear graphs. Suppose ULin is the one-colored pasting scheme of unital linear graphs, so a ULin-prop in \( M \) is a monoid. Each of the \( n \) vertices in the one-colored linear graph \( L_n \) is a tunnel.

For example, consider the linear graph \( L_3 \) with three vertices \( \{1, 2, 3\} \). Suppose \( P \) is a monoid in \( M \). Below we will write \( P_i \) for a copy of \( P \) indexed by a vertex \( i \). Then we have:

\[
\begin{align*}
P[L_3] &= P_1 \otimes P_2 \otimes P_3; \\
P[L_3/\{i\}] &= \bigotimes_{k \neq i} P_k \quad \text{for} \ i \in \{1, 2, 3\}; \\
P[L_3/\{i,j\}] &= P_k \quad \text{where} \ \{i,j,k\} = \{1, 2, 3\}; \\
P[L_3/\{1,2,3\}] &= 1.
\end{align*}
\]

These objects can be organized into a 3-cube in \( M \):

\[
\begin{array}{c}
P_1 \\
P_3 \\
P_1 \otimes P_3 \\
\end{array}
\begin{array}{ccc}
\xrightarrow{P} & \xrightarrow{\text{P}^{-1}} & \xleftarrow{\text{P}^{-1}} \\
\xrightarrow{P} & \xleftarrow{1} \\
\end{array}
\begin{array}{c}
P_2 \\
P_2 \otimes P_3 \\
\end{array}
\begin{array}{c}
P_1 \otimes P_2 \\
P_1 \otimes P_2 \otimes P_3 \\
\end{array}
\]

in which each map is induced by the unit \( 1 : 1 \to P \). The object \( P^{-1}[L_3] \) in (5.2.3) is a colimit of the punctured 3-cube, i.e., a colimit of the diagram consisting of the three faces not containing the terminal vertex \( P_1 \otimes P_2 \otimes P_3 \). The map \( \alpha_G \) in (5.2.3) is the induced map from this colimit to the terminal vertex.

Similarly, for other one-colored linear graphs \( L_n \), the object \( P^{-1}[L_n] \) is a colimit of the punctured \( n \)-cube, i.e., the \( n \)-cube without the terminal vertex.

Motivation 5.2.6. Continuing the setting of 5.2.1, we next consider another way in which a graph \( G \in \mathcal{G}_{n+1}^{(\ell)} \) with \( n + 1 \) ordinary internal edges is related to the sub-category \( \mathcal{G}_n^{(\ell)} \). Recall that the purpose of this consideration is to relate two consecutive filtration strata \( W(\mathcal{G}, J, P)_n \) and \( W(\mathcal{G}, J, P)_{n+1} \) for a \( \mathcal{G} \)-prop \( P \) in \( M \). Here we consider a map

\[
(H_v) : \ G = K(H_v) \to K
\]

in the substitution category \( \mathcal{G}_n^{(\ell)} \). Suppose \( H_v \) is an ordinary graph that has at least one internal edge for some vertex \( x \) in \( K \), and \( H_t \) is a corolla for every other vertex \( t \) in \( K \). Since the ordinary internal edges in \( H_x \) must become ordinary internal edges in \( G = K(H_v) \), there are
strictly more ordinary internal edges in $G$ than in $K$, i.e., $|K| \leq n$. For example, $H_x$ below is an ordinary graph with two internal edges.

So we have a map $(H_x) : G \to K$ in the substitution category $G_2^{(2)}$ with $|K| < |G|$. In the next definition, we will consider graph substitution as above, via a device called the decomposition category of $G$, together with the functor induced by the commutative segment $J$. The ordinary internal edges in $K$ are decorated by $J$, and the ordinary internal edges in each $H_x$ are given length 0.

Also in the next definition is a similar category called the positive decomposition category of $G$. In this category, the internal edges in $K$ are given length 1, while the internal edges in each $H_x$ are decorated by $J$.

These decompositions of $G$ model the $G$-prop structure maps in the $W$-construction. The positive decomposition category will later be used to prove the cofibrancy of the $W$-construction.

**Definition 5.2.7.** Suppose $(\xi \lambda)$ is a pair of $C$-profiles.

1. Define the ordinary substitution category $G_2^{\text{ord}(\xi \lambda)}$ (resp., $G_n^{\text{ord}(\xi \lambda)}$) as the wide sub-category of the substitution category $G_2^{(\xi \lambda)}$ in Def. 3.1.2 (resp., of $G_n^{(\xi \lambda)}$ in Def. 5.1.3) consisting of maps $(H_v)$ in which each $H_v$ is an ordinary graph.

2. For $G \in G_n^{(\xi \lambda)}$, define the decomposition category of $G$, written as $D(G)$, as the full sub-category of $(G \downarrow G_n^{\text{ord}(\xi \lambda)})^{\text{op}}$ consisting of objects $(H_v) : G \to K \in G_n^{\text{ord}(\xi \lambda)}$

such that at least one $|H_v| \geq 1$ (which implies $|K| < |G|$).
(3) For \(G \in \mathcal{G}_{\text{ord}}(\mathcal{C})\), define the map

\[
J^*[G] := \lim_{\overset{G = K(\mathcal{H}_v) \in \mathcal{D}(\mathcal{G})}{v \in \mathcal{K}}} J[K] \otimes \bigotimes_{v \in \mathcal{K}} \mathbb{I}[H_v] \xrightarrow{\beta_G J} J[G]
\]

in which for each object \((H_v) : G \rightarrow K\) in \(\mathcal{D}(\mathcal{G})\), the map

\[
J[K] \rightarrow J[G]
\]

is part of the functor \(J\) (Def. 3.2.4).

(4) For \(G \in \mathcal{G}_{\text{ord}}(\mathcal{C})\), define the positive decomposition category of \(G\), written as \(\mathcal{D}^+(\mathcal{G})\), as the full sub-category of \((\mathcal{G} \downarrow \mathcal{G}_{\text{ord}}(\mathcal{C}))\) consisting of objects

\[
(H_v) : G \rightarrow K \in \mathcal{G}_{\text{ord}}(\mathcal{C})
\]

such that \(|K| \geq 1\) (which implies \(\coprod_{v \in K} |H_v| < |G|\)).

(5) For \(G \in \mathcal{G}_{\text{ord}}(\mathcal{C})\), define the pushout square and the unique induced map \(\beta_G^\circ\)

\[
\begin{array}{ccc}
\coprod_{v \in \mathcal{K}} \mathbb{I}[K] & \otimes & \bigotimes_{v \in \mathcal{K}} \mathbb{I}[H_v] \\
\xrightarrow{\otimes_{v \in \mathcal{K}} \mathbb{I}[H_v]} & & \xrightarrow{\otimes_{v \in \mathcal{K}} \mathbb{I}[H_v]} \\
\xrightarrow{\text{pushout}} & & \xrightarrow{\text{pushout}} \\
\lim_{G = K(\mathcal{H}_v) \in \mathcal{D}^+(\mathcal{G})} \mathbb{I}[K] & \otimes & \bigotimes_{v \in \mathcal{K}} J[H_v] \\
\xrightarrow{\beta_G J} & & \xrightarrow{\beta_G J} \\
\xrightarrow{\delta_G^\circ} & & \xrightarrow{\delta_G^\circ} \\
J'[G] & \rightarrow & J[G]
\end{array}
\]

in which \(0, 1 : \mathbb{I} \rightarrow J\) are part of the commutative segment \(J\). The upper left coproduct is indexed by the set of all maps

\[
(H_v) : G \rightarrow K
\]

in \(\mathcal{G}_{\text{ord}}(\mathcal{C})\) with \(|K| \geq 1\) and at least one \(|H_v| \geq 1\).

(6) For \(G \in \mathcal{G}_{\text{ord}}(\mathcal{C})\), define the pushout products

\[
\delta_G^\circ = \alpha_G \circ \beta_G : (J \otimes P)^*[G] \rightarrow (J \otimes P)[G]
\]

\[
\delta_G^\circ = \alpha_G \circ \beta_G^\circ : (J \otimes P)^*[G] \rightarrow (J \otimes P)[G]
\]

in which \(\alpha_G : P[G] \rightarrow P[G]\) is the map in (5.2.3).}

**Lemma 5.2.11.** For each \(\mathcal{C}\)-colored unital pasting scheme \(\mathcal{G}\) and each \(G \in \mathcal{G}_{\text{ord}}(\mathcal{C})\), the diagram (5.2.9) is well defined.

**Proof.** Let us first unravel the colimits. Suppose

\[
(H_v) : G \rightarrow K \quad \text{and} \quad (H'_v) : G \rightarrow K'
\]

are maps in \(\mathcal{G}_{\text{ord}}(\mathcal{C})\) (in particular, \(G = K(H_v) = K'(H'_v)\)) such that at least one \(|H_v| \geq 1\) and at least one \(|H'_v| \geq 1\). A map

\[
(D_u) : \left( G \xrightarrow{(H'_v)} K' \right) \longrightarrow \left( G \xrightarrow{(H_v)} K \right)
\]

in the decomposition category \(\mathcal{D}(\mathcal{G})\) consists of

1. a map \((D_u) : K \rightarrow K'\) in \(\mathcal{G}_{\text{ord}}(\mathcal{C})\),
2. a partition \(\{H_v : v \in K\} = \coprod_{v \in K} \{H_{uv}\}\), and
(3) a map \((H_u) : H_u' \to D_u \in \bar{G}^{\text{ord}}(u)\) for each vertex \(u \in K'\). In particular, such a map \((D_u) \in D(G)\) determines decompositions
\[
K = K'(D_u) \quad \text{and} \quad H_u' = D_u(H_u)
\]
in which the second decomposition holds for each vertex \(u \in K'\). The map
\[
J[K'] \otimes \bigoplus_{u \in K'} \mathbb{1}[H_u'] \to J[K] \otimes \bigoplus_{v \in K} \mathbb{1}[H_v]
\]
needed to define the colimit \(J[G]\) is the tensor product of copies of the map \(0 : \mathbb{1} \to J\) indexed by the set \(\bigsqcup_{u \in K'} |D_u|\).

Similarly, the lower left colimit over \(D^\circ(G)\) is well defined. Here for each map
\[
(D_u) : ( G \xrightarrow{(H_u)} K ) \to ( G \xrightarrow{(H_u')} K' )
\]
in the positive decomposition category \(D^\circ(G)\), which yields the same decompositions \([5.2.13]\), the map
\[
\mathbb{1}[K] \otimes \bigoplus_{v \in K} J[H_v] \to \mathbb{1}[K'] \otimes \bigoplus_{u \in K'} J[H_u']
\]
needed to define the colimit over \(D^\circ(G)\) is the tensor product of copies of \(1 : \mathbb{1} \to J\) indexed by the set \(\bigsqcup_{u \in K'} |D_u|\).

Since the outermost diagram in \([5.2.9]\) is commutative, the existence and uniqueness of the map \(\beta_G\) follow from the definition of the square as a pushout. \(\square\)

**Example 5.2.15.** This example illustrates the decomposition category of a graph and the map \(\beta_G\) in \([5.2.8]\). Consider the \(\mathcal{C}\)-colored pasting scheme \(\mathcal{G}_\mathcal{C}^\mathbb{Q}\) of connected wheeled graphs and the graph

\[
G \quad \begin{tikzpicture}
\node (u) at (0,0) [shape=circle,draw] {u};
\node (v) at (0,-2) [shape=circle,draw] {v};
\node (e) at (-1,0) [shape=circle,draw] {e};
\node (f) at (1,0) [shape=circle,draw] {f};
\draw (u) edge (e);
\draw (u) edge (f);
\draw (e) edge (v);
\draw (f) edge (v);
\end{tikzpicture}
\]

with two vertices \(\{u, v\}\) and two ordinary internal edges \(\{e, f\}\). The flags in \(G\) can be colored arbitrarily. The decomposition category \(\mathcal{D}(G)\) contains the following objects and non-identity maps:

\[
\begin{array}{ccc}
K_1(H_1) & \leftrightarrow & K_0(H_0) & \to & K_2(H_2)
\end{array}
\]

with two vertices \(\{w, v\}\) and two ordinary internal edges \(\{s, f\}\). The flags in \(G\) can be colored arbitrarily. The decomposition category \(\mathcal{D}(G)\) contains the following objects and non-identity maps:
In the above picture, we abbreviated an object
\[(H_\nu) : G = K(H_\nu) \rightarrow K\]
in the decomposition category to \(K(H_\nu)\). In each object, \(K\) must have the same graph profile as \(G\).

- In the object \(K_0(H_0)\), \(K_0\) is a corolla with one input and one output, and \(H_0\) is \(G\).
- In the object \(K_1(H_1)\), \(K_1\) is a linear graph whose only internal edge is \(f\), and \(H_\nu\) is a corolla, which is not drawn in the picture above. The graph \(H_w\) contains one vertex \(u\) and a loop \(e\).
- In the object \(K_2(H_2)\), \(K_2\) has one vertex \(w\) and a loop \(e\), and \(H_\nu\) is a corolla, which is not drawn in the picture above. The graph \(H_w\) contains one vertex \(u\) and a loop \(e\).

In particular, the inclusion
\[
\begin{align*}
\{ K_1(H_1) & \leftarrow K_0(H_0) \rightarrow K_2(H_2) \} \rightarrow D(G)
\end{align*}
\]
is an equivalence of categories.

Below we will write \(J_e\) to denote a copy of the commutative segment \(J\) indexed by the internal edge \(e\), and similarly for \(J_f\), \(1_e\), and \(1_f\). So we have
\[
\begin{align*}
J[G] &= J_f \otimes J_e; \\
J[K_0] \otimes 1[H_0] &\cong 1_f \otimes 1_e; \\
J[K_1] \otimes 1[H_1] &= J_f \otimes 1_e; \\
J[K_2] \otimes 1[H_2] &\cong 1_f \otimes J_e.
\end{align*}
\]
The map \(\beta^{-}_G\) in \(5.2.8\) is the unique induced dotted map in the diagram
\[
(5.2.16)
\]
in which the square is a pushout.

For instance, suppose \(M = \text{Top}\) and \(J = [0, 1]\). Then \(J[G]\) is the square \([0, 1]^2\), and \(J^{-}[G]\) is the bottom-left boundary
\[
J^{-}[G] = [0, 1] \times \{0\} \cup \{0\} \times [0, 1]
\]
with the point \((0, 0)\) identified. The map \(\beta^{-}_G\) is the bottom-left boundary inclusion into the square.
Example 5.2.17. This example illustrates the positive decomposition category of a graph and the map $\beta^G_\epsilon$ in (5.2.9), using the same pasting scheme $G_{\epsilon \bigcirc}$ and the same graph $G$ as in Example 5.2.15. The positive decomposition category $\mathcal{D}^\diamond(G)$ contains the following objects and non-identity maps:

\[
K_1(H_1) \leftarrow G(C_u,C_v) \longrightarrow K_2(H_2)
\]

- The decompositions $K_1(H_1)$ and $K_2(H_2)$ are the same as in Example 5.2.15. In particular, there are actually four objects in the isomorphism class of $K_2(H_2)$ in $\mathcal{D}^\diamond(G)$, given by permuting the vertex listing at $w$ in $K_2$ and the corresponding graph profile of $H_w$.
- In the object $G(C_u,C_v)$, each $C$ denotes a corolla. As for $K_2(H_2)$, there are four objects in the isomorphism class of $G(C_u,C_v)$, given by permuting the vertex listing at $u$ in $G$ and substituting in a corresponding permuted corolla instead of $C_u$. Our construction below will respect such isomorphisms, so we only need one such representative in the isomorphism class of $G(C_u,C_v)$.

In particular, the inclusion

\[
\left\{ K_1(H_1) \leftarrow G(C_u,C_v) \longrightarrow K_2(H_2) \right\} \longrightarrow \mathcal{D}^\diamond(G)
\]

is an equivalence of categories.

The colimit in the middle left entry in (5.2.9) is the pushout in the diagram:

\[
\begin{array}{cccc}
1_f \otimes 1_e & \xrightarrow{(1,1d)} & J_f \otimes 1_e \\
\downarrow & & \downarrow \\
\text{pushout} & & \text{pushout} \\
1_f \otimes J_e & \xrightarrow{\text{colim}} & \text{colim} \\
G=K(H_e) \in \mathcal{D}^\diamond(G) & & v \in K \\
\downarrow & & \downarrow \\
\text{colim} \otimes 1 & & J[G] = J_f \otimes J_e \\
\downarrow & & \downarrow \\
\{1\} \times [0,1] \cup [0,1] \times \{1\} & & \{1\} \times [0,1] \cup [0,1] \times \{1\}
\end{array}
\]

For instance, suppose $M = \text{Top}$ and $J = [0,1]$. Then $J[G]$ is the square $[0,1] \times [0,1]$, and the pushout in the previous diagram is the right-top boundary

\[
\{1\} \times [0,1] \cup [0,1] \times \{1\}
\]

with the point $(1,1)$ identified. The map $\otimes_{|K|}1$ is the inclusion of the right-top boundary into the square.

Among the decompositions of $G$ above, up to isomorphisms only $K_1(H_1)$ and $K_2(H_2)$ satisfy $|K| \geq 1$ and at least one $|H| \geq 1$. The pushout $J^\diamond[G]$ in (5.2.9) is
isomorphic to the colimit of the solid-arrow outer square in the diagram:

\[
\begin{array}{ccc}
J_f \otimes 1_e & \xrightarrow{(0,1d)} & J_f \otimes J_e \\
(1,1d) \downarrow & & \downarrow (1d,0) \\
1_f \otimes 1_e & \rightarrow & 1_f \otimes 1_e \\
(1d,0) \downarrow & & (1d,1) \\
1_f \otimes J_e & \xleftarrow{(1d,1)} & 1_f \otimes J_e \\
\end{array}
\]

In the diagram (5.2.19):
- The pushout of the top row is \( J^-[G] \) in (5.2.16).
- The pushout of the bottom row is the pushout in (5.2.18).
- The vertical maps correspond to the top and the left maps in (5.2.9).

The map
\[
\beta^*_G : J^o[G] \longrightarrow J[G]
\]
in (5.2.9) is induced by the dotted maps in (5.2.19).

For instance, suppose \( M = \text{Top} \) and \( J = [0,1] \). Then \( J[G] \) is the square \([0,1]^2\), and \( J^o[G] \) is the boundary of the square. The map \( \beta^*_G \) is the boundary inclusion.

### 5.3. Reduction Maps

**Definition 5.3.1.** Suppose \( n \) is a non-negative integer.

1. Define \( \mathcal{G}_n(\rangle) \) as the sub-category of \( \mathcal{G}(\rangle) \) consisting of:
   - objects \( G \) with \( |G| = n \);
   - maps
     \[
     (\sigma_vC_{v'}\tau_v) : G(\sigma_vC_{v'}\tau_v) \longrightarrow G,
     \]
     where each vertex \( v \in G \), \( \sigma_vC_{v'}\tau_v \) is a permuted corolla whose profile agrees with that of \( v \).
   - Note that \( \mathcal{G}_n(\rangle) \) is a groupoid.

2. For an object \( G \in \mathcal{G}_n(\rangle) \), define \( [G]_w \) as the maximal connected subgroupoid of \( \mathcal{G}_n(\rangle) \) containing \( G \).

**Motivation 5.3.2.** The reduction map in the next observation is the actual manifestation of the discussion in Motivation 5.2.1 and Motivation 5.2.6. The point is to relate a graph \( G \) with \( n + 1 \) ordinary internal edges with a graph with strictly fewer ordinary internal edges via a map

\[
(\uparrow_t) : G(\uparrow_t) \longrightarrow G \quad \text{or} \quad (H_v) : G = K(H_v) \longrightarrow K
\]
in the substitution category. The following reduction map will allow us to obtain a tight relationship between two consecutive filtration strata of the \( W \)-construction, which will be used to prove the cofibrancy of the \( W \)-construction.

A pasting scheme is **connected** if each of its graphs is connected.

**Lemma 5.3.3.** For each \( n \geq 0 \), each pair \( (\rangle) \) of \( \mathcal{C} \)-profiles, and each \( \mathcal{G} \)-prop \( P \) in \( M \) with \( \mathcal{G} \) a \( \mathcal{C} \)-colored connected unital pasting scheme, the functors \( P \) (Def. 3.1.2)
and $J$ (Def. 3.2.4) induce a map
\[
\lim_{\mathcal{G} \in \mathcal{G}_{n+1}} (J \otimes P)^{-1}[G] = \bigsqcup_{w \in \mathcal{G}_{n+1}} \left[ \lim_{\mathcal{G} \in \mathcal{G}_{w}} (J \otimes P)^{-1}[G] \right] \xrightarrow{\rho} W(G, J, P)_{n}(\varphi).
\]

**Proof.** Let us first unravel the colimit. Since $\mathcal{G}_{w} \in \mathcal{G}_{n+1}$ is a groupoid, it splits as a coproduct of its maximal connected sub-groupoids. If $(\sigma C \tau) : G' \to G$ is a map in $[G]_{w}$, then $G'$ is obtained from $G$ by reordering the listings at the vertices using $(\tau^{-1}; \sigma^{-1})$, so they have the same tunnels and the same ordinary internal edges. There is a commutative diagram
\[
\begin{array}{c}
J^-[G'] \otimes P^-[G'] \xrightarrow{(\text{Id}, \alpha_{G'})} J^-[G'] \otimes P[G'] \\
J[G'] \otimes P^-[G'] \xrightarrow{(\beta_{G}, \text{Id})} J[G] \otimes P^-[G] \xrightarrow{(\gamma_{P})} J^-[G] \otimes P[G] \\
J^-[G] \otimes P^-[G] \xrightarrow{(\beta_{G}, \text{Id})} J[G] \otimes P^-[G] \\
\end{array}
\]
in which each $\gamma_{P}$ is induced by the $\mathcal{G}$-prop structure map of $P$ at the permuted corollas $\sigma C \tau$. The induced map from the pushout of the back to the pushout of the front is the map
\[
(J \otimes P)^{-1}[G'] \to (J \otimes P)^{-1}[G]
\]
needed to define $\lim_{\mathcal{G} \in \mathcal{G}_{w}} (J \otimes P)^{-1}[G]$.

To define the restriction of the desired map $\rho$ to this colimit, we first define a map
\[
\begin{array}{c}
(J \otimes P)^{-1}[G] \xrightarrow{h_{\alpha}} W(G, J, P)_{n}(\varphi) \\
(J^-[G] \otimes P^-[G]) \xrightarrow{(\text{Id}, \alpha_{G})} J^-[G] \otimes P[G] \\
J[G] \otimes P^-[G] \xrightarrow{(\beta_{G}, \text{Id})} J[G] \otimes P^-[G] \\
\end{array}
\]
for each $G \in \mathcal{G}_{w} \in \mathcal{G}_{n+1}$. By definition there is a pushout diagram
\[
\begin{array}{c}
J^-[G] \otimes P^-[G] \xrightarrow{(\text{Id}, \alpha_{G})} J^-[G] \otimes P[G] \\
J[G] \otimes P^-[G] \xrightarrow{(\beta_{G}, \text{Id})} J[G] \otimes P^-[G] \\
\end{array}
\]
so we first define a map to $W(G, J, P)_{n}(\varphi)$ from each of $J[G] \otimes P^-[G]$ and $J^-[G] \otimes P[G]$. 

5.3. REDUCTION MAPS

To define a map from \( J[G] \otimes P[G] \), consider non-empty subsets of tunnels \( T \subseteq T' \) in \( G \) (Def. 5.2.2). There is a commutative diagram

\[
\begin{array}{ccc}
G/T' & \xrightarrow{(1)_{T',T}} & G/T \\
\downarrow{(1)_T} & & \downarrow{(1)_T} \\
G & & G
\end{array}
\]

in \( \mathcal{G}_{n+1}(\mathcal{G}) \). The top horizontal arrow is in \( \mathcal{G}_{n}(\mathcal{G}) \) because

\[ |G/T'|, |G/T| < |G| = n + 1 \]

when \( \mathcal{G} \) is connected. (If \( G \) is not connected, then it may have a connected component made up of a corolla whose unique vertex is a tunnel, in which case substituting an exceptional edge into that tunnel does not change the number of ordinary internal edges.) This implies that in the diagram

\[
\begin{array}{ccc}
J[G] \otimes P[G/T'] & \xrightarrow{J} & (J \otimes P)[G/T'] \\
\downarrow{\otimes^1_{\gamma^p}J} & & \downarrow{\otimes^1_{\gamma^p}J} \\
J[G/T] \otimes P[G/T'] & \xrightarrow{J} & (J \otimes P)[G/T]
\end{array}
\]

the upper left triangle is commutative. The left trapezoid is commutative because

\[ |G/T'|, |G/T| < |G| = n + 1 \]

and

\[ |G| \leq n \]

in \( \mathcal{G}_{n}(\mathcal{G}) \). The top horizontal arrow is in \( \mathcal{G}_{n}(\mathcal{G}) \) because

\[ |G/T'|, |G/T| < |G| = n + 1 \]

when \( \mathcal{G} \) is connected. (If \( G \) is not connected, then it may have a connected component made up of a corolla whose unique vertex is a tunnel, in which case substituting an exceptional edge into that tunnel does not change the number of ordinary internal edges.) This implies that in the diagram

\[
\begin{array}{ccc}
J[G] \otimes P[G/T'] & \xrightarrow{f_G} & W(\mathcal{G}, J, P)_{n}(\mathcal{G}) \\
\downarrow{\otimes^1_{\gamma^p}J} & & \downarrow{\otimes^1_{\gamma^p}J} \\
J[G/T] \otimes P[G/T'] & \xrightarrow{J} & (J \otimes P)[G/T]
\end{array}
\]

the upper left triangle is commutative. The left trapezoid is commutative by inspection. The right square is commutative by the definition of \( W(\mathcal{G}, J, P)_{n}(\mathcal{G}) \) as a coend over \( \mathcal{G}_{n}(\mathcal{G}) \). So there is a unique map

\[ (5.3.7) \quad J[G] \otimes P[G] \xrightarrow{f_G} W(\mathcal{G}, J, P)_{n}(\mathcal{G}) \]

whose restriction to each \( J[G] \otimes P[G/T] \) for a non-empty subset of tunnels \( T \) in \( G \) is the bottom horizontal composite in the previous diagram.

Next, to define a map from \( J[G] \otimes P[G] \) to \( W(\mathcal{G}, J, P)_{n}(\mathcal{G}) \), consider a map

\[ (D_u) : (G \xrightarrow{\phi^u} K') \rightarrow (G \xrightarrow{\phi^u} K) \]

in the decomposition category \( \mathcal{D}(G) \) as in (5.2.12). Since \( |H_u| \geq 1 \) for at least one vertex \( u \in K' \) and \( |H_u'| \geq 1 \) for at least one vertex \( v \in K \), we have

\[ |K|, |K'| < |G| = n + 1. \]

In the diagram

\[
\begin{array}{ccc}
J[K'] \otimes P[G] & \xrightarrow{\otimes^1_{\gamma^{H_u'}}J} & (J \otimes P)[K'] \\
\downarrow{\otimes^1_{\gamma^{H_u'}}J} & & \downarrow{\otimes^1_{\gamma^{H_u'}}J} \\
J[K] \otimes P[G] & \xrightarrow{J} & (J \otimes P)[K]
\end{array}
\]

the upper left triangle is commutative. The left trapezoid is commutative because

\[ |G/T'|, |G/T| < |G| = n + 1 \]

when \( \mathcal{G} \) is connected. (If \( G \) is not connected, then it may have a connected component made up of a corolla whose unique vertex is a tunnel, in which case substituting an exceptional edge into that tunnel does not change the number of ordinary internal edges.) This implies that in the diagram

\[
\begin{array}{ccc}
J[G] \otimes P[G] & \xrightarrow{J} & (J \otimes P)[G] \\
\downarrow{\otimes^1_{\gamma^p}J} & & \downarrow{\otimes^1_{\gamma^p}J} \\
J[K] \otimes P[G] & \xrightarrow{J} & (J \otimes P)[K]
\end{array}
\]

the upper left triangle is commutative. The left trapezoid is commutative by inspection. The right square is commutative by the definition of \( W(\mathcal{G}, J, P)_{n}(\mathcal{G}) \) as a coend over \( \mathcal{G}_{n}(\mathcal{G}) \). So there is a unique map

\[ (5.3.7) \quad J[G] \otimes P[G] \xrightarrow{f_G} W(\mathcal{G}, J, P)_{n}(\mathcal{G}) \]

whose restriction to each \( J[G] \otimes P[G/T] \) for a non-empty subset of tunnels \( T \) in \( G \) is the bottom horizontal composite in the previous diagram.
the top left triangle is commutative by the decompositions (5.2.13)

\[ K = K'(D_u) \quad \text{and} \quad H'_u = D_u(H_u). \]

The lower left trapezoid is commutative by inspection. The right square is commutative by the definition of \( W(\mathcal{G}, J, P)_{n}(\frac{\partial}{\partial}) \) as a coend over \( \mathcal{G}_{n}(\frac{\partial}{\partial}) \). So there is a unique map

\[ J^{-}[G] \otimes P[G] \xrightarrow{g_G} W(\mathcal{G}, J, P)_{n}(\frac{\partial}{\partial}) \]

whose restriction to each \( J[K] \otimes P[G] \), with \( (H_u) : G \rightarrow K \) an object in \( D(G) \), is the bottom horizontal composite in the previous diagram.

To see that \( f_G \) (5.3.7) and \( g_G \) (5.3.8) together determine a unique map (5.3.5), by the pushout diagram (5.3.6), we must show that the diagram

\[
\begin{align*}
J^{-}[G] \otimes P^{-}[G] & \xrightarrow{(\beta_G, 1d)} J^{-}[G] \otimes P[G] \\
\downarrow & \downarrow g_G \\
J[G] \otimes P^{-}[G] & \xrightarrow{f_G} W(\mathcal{G}, J, P)_{n}(\frac{\partial}{\partial})
\end{align*}
\]

is commutative. Suppose \( T \) is a non-empty subset of tunnels in \( G \) and \( (H_v) : G \rightarrow K \) an object in the decomposition category \( D(G) \). It is enough to show that the solid-arrow diagram

\[ J[K] \otimes P[G/T] \xrightarrow{\gamma_K} J[K] \otimes P[G] \xrightarrow{\gamma_K \otimes \gamma_{\mathcal{P}}(t)} (J \otimes P)[K] \]

\[ J[G] \otimes P[G/T] \xrightarrow{\gamma_{\mathcal{P}}(t)} (J \otimes P)[G/T] \xrightarrow{\omega_{G/T}} W(\mathcal{G}, J, P)_{n}(\frac{\partial}{\partial}) \]

is commutative. By assumption we have decompositions

\[ G/T = G((t)T) = [K(H_v)]((t)T) = K([(H_v)(T_v)]) \]

in which \( T_v \subseteq T \) is the subset of tunnels in \( H_v \). This implies that the left triangle and the top wedge in (5.3.9) are commutative. Since there is a map

\[ G/T - \left( (H_v)(T_v) \right) \rightarrow K \]

in \( \mathcal{G}_{n}(\frac{\partial}{\partial}) \), the trapezoid in the above diagram is also commutative by the definition of \( W(\mathcal{G}, J, P)_{n}(\frac{\partial}{\partial}) \) as a coend over \( \mathcal{G}_{n}(\frac{\partial}{\partial}) \). We have shown that the solid-arrow diagram in (5.3.9) is commutative, so there is a unique dotted arrow \( h_G \)

\[
\begin{align*}
J^{-}[G] \otimes P^{-}[G] & \xrightarrow{(\beta_G, 1d)} J^{-}[G] \otimes P[G] \\
\downarrow & \downarrow g_G \\
J[G] \otimes P^{-}[G] & \xrightarrow{f_G} W(\mathcal{G}, J, P)_{n}(\frac{\partial}{\partial})
\end{align*}
\]
that makes the diagram commutative.

Finally, we claim that the maps \( h_\mathcal{G} \) induce a map

\[
(5.3.10) \quad \text{colim}_{\mathcal{G} \in \mathcal{G}_w} (J \otimes P)^{-\mathcal{G}} \xrightarrow{\eta_\mathcal{G} T} W(\mathcal{G}, J, P)_{n, w}^{(\frac{3}{2})}
\]

for each maximal connected sub-groupoid \([\mathcal{G}]_w\) in \(\mathcal{G}_{n+1}^{(\frac{3}{2})}\). Suppose

\[
(\sigma_v C_v \tau_v) : \mathcal{G}' = G(\sigma_v C_v \tau_v) \longrightarrow G
\]
is a map in \([\mathcal{G}]_w\). We must show that the diagram

\[
(5.3.11) \quad (J \otimes P)^{-\mathcal{G}'} \xrightarrow{h_{\mathcal{G}'}} W(\mathcal{G}, J, P)_{n, w}^{(\frac{3}{2})}
\]
is commutative. It suffices to show that the diagrams

\[
(5.3.12)
\]
\[
\begin{array}{ccc}
J[G'] \otimes P^{-\mathcal{G}'} & \xrightarrow{f_{\mathcal{G}'}} & W(\mathcal{G}, J, P)_{n, w}^{(\frac{3}{2})} \\
(z, \gamma^P) & \downarrow & \\
J[G] \otimes P^{-\mathcal{G}} & \xrightarrow{f_{\mathcal{G}}} & W(\mathcal{G}, J, P)_{n, w}^{(\frac{3}{2})}
\end{array} =
\begin{array}{ccc}
J^{-\mathcal{G}'} \otimes P[G'] & \xrightarrow{g_{\mathcal{G}'}} & W(\mathcal{G}, J, P)_{n, w}^{(\frac{3}{2})} \\
\gamma^P & \downarrow & \\
J^{-\mathcal{G}} \otimes P[G] & \xrightarrow{g_{\mathcal{G}}} & W(\mathcal{G}, J, P)_{n, w}^{(\frac{3}{2})}
\end{array}
\]

are commutative.

For the left diagram, suppose \( T \) is a non-empty subset of tunnels in \( \mathcal{G}' \), which implies that \( T \) is also a non-empty subset of tunnels in \( \mathcal{G} \). There is an induced map

\[
(\sigma_v C_v \tau_v) : \mathcal{G}'/T \longrightarrow \mathcal{G}/T
\]
in \( \mathcal{G}_{n}^{(\frac{3}{2})} \), where \( v \) runs through the set of vertices in \( \mathcal{G} \) that are not in \( T \).

In the diagram

\[
\begin{array}{ccc}
J[G'] \otimes P[\mathcal{G}'/T] & \xrightarrow{J} & (J \otimes P)[\mathcal{G}'/T] \\
(z, \gamma^P) & \downarrow & \\
J[\mathcal{G}/T] \otimes P[\mathcal{G}'/T] & \xrightarrow{\gamma^P} & (J \otimes P)[\mathcal{G}/T]
\end{array}
\]

\[
\begin{array}{ccc}
\xrightarrow{\omega_{\mathcal{G}'/T}} & \xrightarrow{\omega_{\mathcal{G}/T}} & \\
W(\mathcal{G}, J, P)_{n, w}^{(\frac{3}{2})} & \xrightarrow{\omega_{\mathcal{G}/T}} & W(\mathcal{G}, J, P)_{n, w}^{(\frac{3}{2})}
\end{array}
\]

each structure map \( \gamma^P \) is an equivariant structure map of the form \( \gamma^P_{\mathcal{G}, \mathcal{G}'}, \tau_v \), and both maps in the middle column are isomorphisms. The left square is commutative by inspection, and the right square is commutative by the definition of \( W(\mathcal{G}, J, P)_{n, w}^{(\frac{3}{2})} \) as a coend over \( \mathcal{G}_{n}^{(\frac{3}{2})} \). This implies that the left diagram in \( (5.3.12) \) is commutative.
For the right diagram in (5.3.12), suppose
\[(H_u) : G' \longrightarrow K \in \mathcal{G}^{\text{ord}}_{n+1}(\mathcal{L})\]
is an object in the decomposition category \(\mathcal{D}(G')\). Since
\[G' = K(H_u) = G(\sigma_v C_v \tau_v),\]
we have that
\[G = K(H'_u) \quad \text{and} \quad H_u = H'_u (\sigma_v C_v \tau_v),\]
in which \(v\) runs through the vertices in \(G\) corresponding to vertices in \(H_u\). In the diagram
\[
\begin{align*}
\xymatrix@C-2cm{J[K] \otimes P[G'] \ar[r]^-{\gamma_{H_u}^P} \ar[d]_-{\gamma^P} & (J \otimes P)[K] \ar[r]^-{\omega_{K}} & W(G, J, P)_{n}(\frac{d}{2}) \ar[d]^-{=} \\
J[K] \otimes P[G] \ar[r]^-{\gamma_{H'_u}^P} & (J \otimes P)[K] \ar[r]^-{\omega_{K}} & W(G, J, P)_{n}(\frac{d}{2})}
\end{align*}
\]
the left vertical map \(\gamma^P\) is \(\bigotimes_v \gamma^P_{\sigma_v C_v \tau_v}\), so the left square is commutative by the associativity of the \(G\)-prop structure map of \(P\). The right square is commutative by definition. This implies that the right diagram in (5.3.12) and hence the diagram (5.3.11) are commutative. We have defined the map \(\rho[G]_{w}\) in (5.3.10), which is the desired map \(\rho\) when restricted to a coproduct summand corresponding to a maximal connected sub-groupoid \([G]_w\) in \(\mathcal{G}^{\text{ord}}_{n+1}(\mathcal{L})\).

**Example 5.3.13.** To illustrate the reduction map \(\rho\) in Lemma 5.3.3, recall from the above proof that \(\rho\) is defined by the maps
\[h_G : (J \otimes P)^{-}[G] \longrightarrow W(G, J, P)_{n}(\frac{d}{2})\]
in (5.3.5) with \(|G| = n + 1\). Each such map \(h_G\) is in turn uniquely induced by the maps
\[
\begin{align*}
J^{-}[G] \otimes P[G] & \ar[d]^-{g_G} \\
J[G] \otimes P^{-}[G] & \ar[r]^-{f_G} & W(G, J, P)_{n}(\frac{d}{2})
\end{align*}
\]
in (5.3.7) and (5.3.8). We will explain these two maps for the graph
\[
\begin{tikzpicture}
\node (v) at (0,0) {$v$};
\node (a) at (-1,1) {$a$};
\node (b) at (1,1) {$b$};
\node (u) at (0,2) {$u$};
\draw (v) edge (u)
\end{tikzpicture}
\]
in the \(\mathcal{C}\)-colored pasting scheme \(\mathcal{G}\) of connected wheeled graphs, which we introduced in Example 5.2.15. The colors of the legs and the internal edges are as indicated. Suppose \(P\) is a \(\mathcal{C}\)-colored wheeled properad in \(\mathcal{M}\).
The graph $G$ has two ordinary internal edges and one tunnel $v$, which connects an input leg and an internal edge. So we have

$$P^*[G] = \colim_{T \in \text{Tun}(G)} P[G/T] = P[G(\uparrow_c)] = P(u),$$

where the exceptional edge $\uparrow_c$ is substituted into $v$.

The map $f_G$ is the composite

$$J_b \otimes J_c \otimes P(u) \xrightarrow{(\text{Id}, \epsilon, \text{Id})} J_b \otimes \text{I} \otimes P(u) \cong (J \otimes P)[G(\uparrow_c)]$$

in which $\epsilon : J_c \to \text{I}$ is the counit of the commutative segment $J$. As before we write $J_b$ for a copy of $J$ indexed by the internal edge with color $b$, and similarly for $J_c$.

For the map $g_G$, recall from Example 5.2.15 that the decomposition category $D(G)$ is equivalent to the category

$$K_1(H_1) \hookrightarrow K_0(H_0) \to K_2(H_2)$$

and that $J^-[G]$ is the pushout in 5.2.16. Since $- \otimes P[G]$ commutes with colimits, applying it to the pushout 5.2.16 yields the pushout in the following diagram. The map $g_G$ is then the unique induced map in the commutative diagram:

$$
\begin{array}{ccc}
J[K_0] \otimes P[G] & \xrightarrow{\text{pushout}} & J[K_2] \otimes P[G] \\
\downarrow & & \downarrow \\
J[K_1] \otimes P[G] & \xrightarrow{\omega_{K_1}} & J^-[G] \otimes P[G] \\
(\text{Id}, \gamma^P_{H_1}) & & \gamma^P_{K_1} \omega_{K_1} \\
(\text{Id}, \gamma^P_{H_1}) & & \downarrow \gamma^P_{K_1} \\
(\text{Id}, \gamma^P_{H_1}) & & \downarrow \gamma^P_{K_1} \\
(\text{Id}, \gamma^P_{H_1}) & & \downarrow \gamma^P_{K_1} \\
(\text{Id}, \gamma^P_{H_1}) & & \downarrow \gamma^P_{K_1} \\
(\text{Id}, \gamma^P_{H_1}) & & \downarrow \gamma^P_{K_1} \\
(\text{Id}, \gamma^P_{H_1}) & & \downarrow \gamma^P_{K_1} \\
(\text{Id}, \gamma^P_{H_1}) & & \downarrow \gamma^P_{K_1} \\
(\text{Id}, \gamma^P_{H_1}) & & \downarrow \gamma^P_{K_1} \\
(\text{Id}, \gamma^P_{H_1}) & & \downarrow \gamma^P_{K_1} \\
(\text{Id}, \gamma^P_{H_1}) & & \downarrow \gamma^P_{K_1} \\
\end{array}
$$

In the previous diagram, the maps

$$P[G] = P[K_1(H_1)] \xrightarrow{\gamma^P_{H_1}} P[K_1] \quad \text{and} \quad P[G] = P[K_2(H_2)] \xrightarrow{\gamma^P_{H_2}} P[K_2]$$

are structure maps of the wheeled properad $P$. Note that $H_1$ (resp., $H_2$) is a contracted corolla (resp., dioperadic graph), so $\gamma^P_{H_1}$ (resp., $\gamma^P_{H_2}$) is a contraction (resp., dioperadic composition) [Y, 216] (p.216).
5. FILTERING THE BOARDMAN-VOGT CONSTRUCTION

5.4. A Pushout Relating Two Strata

**Motivation 5.4.1.** We are working toward a way to construct the \((n+1)\)st filtration stratum of the \(W\)-construction in terms of the previous filtration stratum. In Theorem 5.4.7 below we will prove that the map

\[
W(\mathcal{G}, J, P)_{n+1} \longrightarrow W(\mathcal{G}, J, P)_{n+1}
\]

is a pushout of the map \(\delta^*\) in Lemma 5.4.2 below along the reduction map \(\rho\) in Lemma 5.3.3 above. Briefly, the map \(\delta^*\) below is induced by the map \(\delta_\mathcal{G}^*\) (5.2.10), which is defined as the pushout corner map in the following diagram.

\[
\begin{array}{ccc}
J^-[G] \otimes P^-[G] & \longrightarrow & J^-[G] \otimes P[G] \\
\downarrow \scriptstyle{(\beta_\mathcal{G}, \text{Id})} & & \downarrow \scriptstyle{(\beta_\mathcal{G}, \text{Id})} \\
J[G] \otimes P^-[G] & \longrightarrow & (J \otimes P)^-[G] \\
\downarrow \scriptstyle{(\text{Id}, \alpha_\mathcal{G})} & & \downarrow \scriptstyle{\delta_\mathcal{G}} \\
(\text{Id}, \alpha_\mathcal{G}) & \longrightarrow & (J \otimes P)[G]
\end{array}
\]

In other words, the map \(\delta^*\) is induced by the map \(0 : 1 \longrightarrow J\) and the colored units of \(P\).

Similarly, the map \(\delta^\triangledown\) below is induced by the map \(\delta_\mathcal{G}^\triangledown\), which is defined as the pushout corner map of \(\alpha_\mathcal{G}\) and \(\beta_\mathcal{G}\). So it is induced by the maps \(0, 1 : 1 \longrightarrow J\) and the colored units of \(P\).

**Lemma 5.4.2.** For each \(n \geq 0\), each pair \((\mathcal{C}, \mathcal{C}')\) of \(\mathcal{C}\)-profiles, and each \(\mathcal{G}\)-prop \(P\) in \(\mathcal{M}\) with \(\mathcal{G}\) a \(\mathcal{C}\)-colored unital pasting scheme, the maps \(\delta_\mathcal{G}^*\) and \(\delta_\mathcal{G}^\triangledown\) (5.2.10) induce a commutative diagram

\[
\begin{array}{ccc}
\colim_{\mathcal{G} \in \mathcal{G}_{n+1}(\mathcal{C}, \mathcal{C}')} (J \otimes P)^-[G] & \longrightarrow & \colim_{\mathcal{G} \in \mathcal{G}_{n+1}(\mathcal{C}, \mathcal{C}')} (J \otimes P)[G] \\
\downarrow \scriptstyle{\xi} & & \downarrow \scriptstyle{=} \\
\colim_{\mathcal{G} \in \mathcal{G}_{n+1}(\mathcal{C}, \mathcal{C}')} (J \otimes P)^\triangledown[G] & \longrightarrow & \colim_{\mathcal{G} \in \mathcal{G}_{n+1}(\mathcal{C}, \mathcal{C}')} (J \otimes P)[G]
\end{array}
\]

with the vertical map \(\xi\) induced by the map \(J^-[G] \longrightarrow J^\triangledown[G]\) in Lemma 5.2.11.

**Proof.** To see that the maps \(\delta_\mathcal{G}^*\) induce the map \(\delta^*\), it suffices to show that, for each map

\[
(\sigma_v C_{\nu', \nu}) : G' \longrightarrow G
\]

in \(\mathcal{G}_{n+1}(\mathcal{C}, \mathcal{C}')\), the induced diagram

\[
\begin{array}{ccc}
(J \otimes P)^-[G'] & \longrightarrow & (J \otimes P)^-[G] \\
\downarrow \scriptstyle{\delta_\mathcal{G}'} & & \downarrow \scriptstyle{\delta_\mathcal{G}} \\
(J \otimes P)[G'] & \longrightarrow & (J \otimes P)[G]
\end{array}
\]

is a pushout. This follows from the fact that \(J^-[G] \longrightarrow J^\triangledown[G]\) is a pushout along \(\rho\) in Lemma 5.3.3 above.
is commutative. The previous diagram is induced by the commutative cube

\[
\begin{array}{c}
J^-[G'] \otimes P^-[G'] \\
\downarrow \beta'_{G'} \\
J^-[G] \otimes P^-[G] \\
\downarrow \beta_G \\
J[G'] \otimes P^-[G'] \\
\downarrow \alpha'_{G'} \\
J[G] \otimes P^-[G] \\
\downarrow \alpha_G \\
(\alpha \otimes P)[G'] \\
\downarrow \beta_{G'} \\
(\beta \otimes P)[G] \\
\end{array}
\]

in which the left and the top faces are from \[5.3.4\], and all four unnamed maps are induced by the \(G\)-prop equivariant structure maps \(\gamma_{\alpha, G'} \circ \tau'\). The top horizontal map in \[5.4.3\] is the induced map from the pushout of the back face to the pushout of the front face of the cube. The commutativity of the square \[5.4.3\] is a consequence of the fact that each \(\beta_G\) factors through \(\beta_{G'}\) by construction \[5.2.9\].

**Example 5.4.4.** As an illustration of the map \(\delta^*\) in Lemma \[5.4.2\] let us consider the map

\[
\delta_G = \alpha_G \circ \beta_G : (J \otimes P)^- G \rightarrow (J \otimes P)[G]
\]

for the graph \(G \in \mathcal{G}_c(\mathcal{C})\) in Example \[5.3.13\] with \(M = \text{Top}\) and \(J = [0, 1]\). Suppose \(P\) is a \(\mathcal{C}\)-colored wheeled properad in \(M\). By Example \[5.2.15\] the map

\[
\beta_G : J^- G \rightarrow J[G]
\]

is the bottom-left boundary inclusion

\[
J^- G = [0, 1] \times \{0\} \cup \{0\} \times [0, 1] \xrightarrow{\beta_G} [0, 1]^2 = J[G]
\]

into the square. From Example \[5.3.13\] \(\alpha_G\) is the map

\[
P^- G = P(u) \times \ast \xrightarrow{\text{Id}_c, 1_c} P(u) \times P(v) = P[G]
\]

in which \(1_c\) is the \(c\)-colored unit of the wheeled properad \(P\). The pushout corner map \(\delta_G = \alpha_G \Box \beta_G\) is the inclusion map

\[
\begin{align*}
\delta_G : J^- (J \otimes P)^- G &\rightarrow J^- G \times \delta G \\
&\downarrow \delta G \\
\end{align*}
\]

\[
(\alpha \otimes P)[G] = [0, 1]^2 \times P(u) \times P(v).
\]
EXAMPLE 5.4.5. As an illustration of the maps \( \delta^\bullet \) and \( \xi \) in Lemma 5.4.2, let us consider the map

\[
\delta_G^\circ = \alpha_G \Box \beta_G^\circ : (J \otimes P)^{\circ}[G] \longrightarrow (J \otimes P)[G]
\]

for the graph \( G \in \text{Gr}_c^G(J) \) in Example 5.3.13 with \( M = \text{Top} \) and \( J = [0, 1] \). Suppose \( P \) is a \( C \)-colored wheeled properad in \( M \). By Example 5.2.17

\[
\beta_G^\circ : J^\circ[G] \longrightarrow J[G]
\]

is the boundary inclusion

\[
J^\circ[G] = [0, 1] \times \{0, 1\} \cup \{0, 1\} \times [0, 1] \overset{\beta_G^\circ}{\longrightarrow} [0, 1]^{\times 2} = J[G]
\]

into the square. The pushout corner map \( \delta_G^\circ = \alpha_G \Box \beta_G^\circ \) is the inclusion map

\[
\begin{split}
(J \times P)^{\circ} &= \left( [0, 1]^{\times 2} \times P(u) \times \ast \right) \cup \left( J^\circ[G] \times P(u) \times P(v) \right) \\
\delta_G^\circ \downarrow \downarrow &
\end{split}
\]

\[
(\delta \times P)[G] = [0, 1]^{\times 2} \times P(u) \times P(v).
\]

The map \( \xi \) is induced by the inclusion

\[
J^{-}[G] = [0, 1] \times \{0\} \cup \{0\} \times [0, 1] \overset{\beta_G^\circ}{\longrightarrow} [0, 1] \times \{0, 1\} \cup \{0, 1\} \times [0, 1] = J^{\circ}[G].
\]

MOTIVATION 5.4.6. For a pasting scheme \( \mathcal{G} \) and a \( \mathcal{G} \)-prop \( P \) in \( M \), recall that the filtration stratum (5.1.4)

\[
W(\mathcal{G}, J, P)_{n+1}(\xi) = \int^{\mathcal{G}(\xi)}_{\mathcal{G}(\xi)} J[G] \otimes P[G]
\]

is defined as a coend over the full sub-category \( \mathcal{G}_{n+1}(\xi) \) of the substitution category \( \mathcal{G}^{(\xi)} \) consisting of \( G \) such that \( |G| \leq n + 1 \). Intuitively, the filtration stratum \( W(\mathcal{G}, J, P)_{n+1}(\xi) \) is the space of decorated graphs \( (J \otimes P)[G] \) with \( |G| \leq n + 1 \) and with relations parametrized by the category \( \mathcal{G}_{n+1}(\xi) \). If \( |G| \leq n + 1 \), then either \( |G| \leq n \), in which case \( G \in \mathcal{G}_{n+1}(\xi) \), or \( |G| = n + 1 \), in which case \( G \in \mathcal{G}_{n+1}(\xi) \). So we should be able to describe the filtration stratum \( W(\mathcal{G}, J, P)_{n+1}(\xi) \) in terms of:

- decorated graphs \( (J \otimes P)[G] \) with \( |G| \leq n \), which is the previous filtration stratum \( W(\mathcal{G}, J, P)_{n}(\xi) \);
- decorated graphs \( (J \otimes P)[G] \) with \( |G| = n + 1 \), which is a colimit over \( \mathcal{G}_{n+1}(\xi) \).

The following observation says that this is indeed possible using the reduction map \( \rho \) and the map \( \delta^\bullet \) above.

THEOREM 5.4.7. For each \( n \geq 0 \), each pair \( (\xi) \) of \( \mathcal{C} \)-profiles, and each \( \mathcal{G} \)-prop \( P \) in \( M \) with \( \mathcal{G} \) a \( \mathcal{C} \)-colored connected unital pasting scheme, there is a pushout diagram
(5.4.8) \[
\colim_{\mathcal{G}_{n+1}^{(\mathbb{Z})}} (J \otimes P)^{-}[G] \xrightarrow{\delta^*} \colim_{\mathcal{G}_{n+1}^{(\mathbb{Z})}} (J \otimes P)[G]
\]
\[
\rho \downarrow
\]
\[
W(\mathcal{G}, J, P)_{n}(\mathbb{Z}) \xrightarrow{} W(\mathcal{G}, J, P)_{n+1}(\mathbb{Z})
\]
such that:

- \(\delta^*\) and \(\rho\) are the maps in Lemmas 5.4.2 and 5.3.3.
- The bottom horizontal map is from Prop. 5.1.15.
- The right vertical map is from the sub-category inclusion \(\mathcal{G}_{n+1}(\mathbb{Z}) \subseteq \mathcal{G}_{n+1}(\mathbb{Z})\).

\textbf{Proof.} We will check that \(W(\mathcal{G}, J, P)_{n+1}(\mathbb{Z})\) has the universal property of a pushout of \(\rho\) and \(\delta^*\). As \(W(\mathcal{G}, J, P)_{n+1}(\mathbb{Z})\) is a coend over \(\mathcal{G}_{n+1}(\mathbb{Z})\), the commutativity of the diagram (5.4.8) follows from those of

\[
\begin{array}{ccc}
\mathcal{G}_{n+1}(\mathbb{Z}) & \xrightarrow{\delta^*} & \mathcal{G}(\mathbb{Z}) \\
J[\mathbb{Z}] \otimes P[G/T] & \xrightarrow{\omega_G} & J[\mathbb{Z}] \otimes P[G] \\
J & \downarrow & (J \otimes P)[G] \\
(J \otimes P)[G/T] & \xrightarrow{\omega_G} & W(\mathcal{G}, J, P)_{n}(\mathbb{Z}) \\
\end{array}
\]

for a non-empty subset of tunnels \(T \in \mathcal{G}_{n+1}(\mathbb{Z})\), which yields a map

\[(\dagger)_T : G/T \xrightarrow{} G\]
in \(\mathcal{G}_{n+1}(\mathbb{Z})\), and of

\[
\begin{array}{ccc}
\mathcal{G}_{n+1}(\mathbb{Z}) & \xrightarrow{\delta^*} & \mathcal{G}(\mathbb{Z}) \\
J[\mathbb{Z}] \otimes P[G] & \xrightarrow{\omega_G} & J[\mathbb{Z}] \otimes P[G] \\
J & \downarrow & (J \otimes P)[G] \\
(J \otimes P)[K] & \xrightarrow{\omega_G} & W(\mathcal{G}, J, P)_{n+1}(\mathbb{Z}) \\
\end{array}
\]

for an object

\[(H_u)_T : G \xrightarrow{} K\]
in the decomposition category \(D(G)\).

Next, suppose given a solid-arrow commutative diagram

\[
\begin{array}{ccc}
\colim_{\mathcal{G}_{n+1}^{(\mathbb{Z})}} (J \otimes P)^{-}[G] & \xrightarrow{\delta^*} & \colim_{\mathcal{G}_{n+1}^{(\mathbb{Z})}} (J \otimes P)[G] \\
\rho \downarrow & & \downarrow \\
W(\mathcal{G}, J, P)_{n}(\mathbb{Z}) & \xrightarrow{} & W(\mathcal{G}, J, P)_{n+1}(\mathbb{Z})
\end{array}
\]
We must show that there exists a unique map $\chi$ that extends both $a$ and $b$. Since an object in $\mathcal{G}_{n+1}(\mathcal{H})$ is either an object in $\mathcal{G}_n(\mathcal{H})$ or an object in $\mathcal{G}_{n+1}(\mathcal{H})$, the only possible choice of such a map $\chi$, when restricted to $(J \otimes P)[G]$ for $G \in \mathcal{G}_{n+1}(\mathcal{H})$, is the restriction of $a$ if $|G| \leq n$ or the restriction of $b$ if $|G| = n + 1$. It remains to show that these definitions indeed define a map from $W(\mathcal{G}, J, P, 0)(\mathcal{H})$, which is a coend over $\mathcal{G}_{n+1}(\mathcal{H})$, to $Y$.

Suppose $(H_v): G \rightarrow K$ is a map in $\mathcal{G}_{n+1}(\mathcal{H})$. To show that we have a well-defined map $\chi$, we must show that the diagram

\[
\begin{array}{ccc}
J[K] \otimes P[G] & \xrightarrow{\otimes_{v \in K} P[H_v]} & (J \otimes P)[K] \\
J \downarrow & & \downarrow q_2 \\
(J \otimes P)[G] & \xrightarrow{q_1} & Y
\end{array}
\]

is commutative, in which the maps $q_1$ and $q_2$ are restrictions of either $a$ or $b$, depending on whether $|G|$ and $|K|$ are at most $n$ or equal to $n + 1$. If both $|G|$ and $|K|$ are at most $n$, then $q_1$ and $q_2$ are both restrictions of $a$, and the above diagram is commutative because $W(\mathcal{G}, J, P, 0)(\mathcal{H})$ is a coend over $\mathcal{G}_{n+1}(\mathcal{H})$. So we may assume that $|G| = n + 1$, $|K| = n + 1$, or both.

Since $\mathcal{G}$ is connected and $n + 1 \geq 1$, each $H_v$ is a permuted corolla, an exceptional edge, or an ordinary graph with $|H_v| \geq 1$. Partitioning the $H_v$ into these three families, we have a factorization of the original map $(H_v): G \rightarrow K$ as

\[
\begin{array}{c}
\begin{array}{c}
G \xrightarrow{(H_u)_{u \in U}} K_2 \xrightarrow{(t)_{t \in T}} K_1 \xrightarrow{(\sigma_x C_x \tau_x)_{x \in X}} K
\end{array}
\end{array}
\]

such that:

- $U \cup T \cup X$ is the set of vertices in $K$.
- Each $H_u$ for $u \in U$ is ordinary with $|H_u| \geq 1$.
- Each $H_t = \uparrow t$ for $t \in T$ is an exceptional edge.
- Each $H_x = \sigma_x C_x \tau_x$ for $x \in X$ is a permuted corolla.

If $U$, $T$, or $X$ is empty, then we simply ignore it, and the proof below will simplify accordingly. In the next paragraph, we will assume that $U$, $T$, and $X$ are all non-empty.

The factorization \[5.4.10\] yields the following diagram, in which $q_3$ and $q_4$ are restrictions of either $a$ or $b$, depending on whether $|K_1|$ and $|K_2|$ are at most $n$ or
equal to \(n+1\).

(5.4.11)

\[
\begin{align*}
J[K] \otimes P[G] & \longrightarrow J[K] \otimes P[K_1] \longrightarrow (J \otimes P)[K] \\
J[K_1] \otimes P[G] & \longrightarrow J[K_1] \otimes P[K_2] \longrightarrow (J \otimes P)[K_1] \\
J[K_2] \otimes P[G] & \longrightarrow (J \otimes P)[K_2] \\
& \longrightarrow Y
\end{align*}
\]

In the diagram (5.4.11):

1. The outermost diagram is the diagram (5.4.9).
2. The top wedge is commutative by the unity and associativity of the \(\mathcal{G}\)-prop structure map \(\gamma^p\). The left wedge is commutative because \(J\) is a functor (Def. 3.2.4).
3. The top left rectangle and the middle left square are commutative by inspection and the unity and associativity of \(\gamma^p\).
4. The right trapezoid involving \(q_2\) and \(q_3\) is commutative because \(q_2\) and \(q_3\) are both restrictions of \(a\) if \(|K| = |K_1| \leq n\), in which case
   \[
   (\sigma_x \mathcal{C}_x \tau_x) : K_1 \longrightarrow K
   \]
   is a map in \(\mathcal{G}_{\leq n}^{(2)}\). If \(|K| = |K_1| = n + 1\), in which case
   \[
   (\sigma_x \mathcal{C}_x \tau_x) : K_1 \longrightarrow K
   \]
   is a map in \(\mathcal{G}_{\leq n+1}^{(2)}\), then \(q_2\) and \(q_3\) are both restrictions of \(b\).
5. The quadrilateral involving \(q_1\) and \(q_4\) is commutative because both \(q_3\) and \(q_4\) are restrictions of \(a\) if \(|K_1| \leq n\) (and hence \(|K_2| \leq n\)), in which case
   \[
   (t_1) : K_2 \longrightarrow K_1
   \]
   is a map in \(\mathcal{G}_{\leq n}^{(4)}\). If \(|K_1| = n + 1\), then this quadrilateral is commutative because \(a \circ b = b \circ^*\) when restricted to \(J[K_1] \otimes P^*[K_1]\).
6. The bottom trapezoid involving \(q_1\) and \(q_4\) is commutative because both \(q_1\) and \(q_4\) are restrictions of \(a\) if \(|G| \leq n\) (and hence \(|K_2| \leq n\)), in which case
   \[
   (H_u)_{\text{weu}} : G \longrightarrow K_2
   \]
   is a map in \(\mathcal{G}_{\leq n}^{(2)}\). If \(|G| = n + 1\), then this trapezoid is commutative because \(a \circ b = b \circ^*\) when restricted to \(J^*[G] \otimes P[G]\).

Therefore, the diagram (5.4.11) is commutative.

**Example 5.4.12.** Let us consider an example of the factorization in (5.4.10), which was a crucial part of the previous proof. Consider the factorization in the substitution category \(\mathcal{G}_c\langle\delta\rangle\) of connected wheeled graphs depicted as follows.
So we have that
\[ K_2 = K(H_{t_1}, H_{t_2}); \]
\[ G = K_2(H_u) = K(H_u, H_{t_1}, H_{t_2}). \]

The ordinary graph \( K \) on the right has three \( c \)-colored internal edges, three vertices \( \{u, t_1, t_2\} \), and two tunnels \( \{t_1, t_2\} \). Both \( H_{t_1} \) and \( H_{t_2} \) are the exceptional edge \( \uparrow_c \), so \( K_2 \) has a single internal edge, which is a \( c \)-colored loop at its only vertex \( u \). The ordinary graph \( H_u \) has two vertices and two internal edges. The ordinary graph \( G \) on the left has two vertices and three internal edges. In this example, the set \( X \) in (5.4.10) is empty.

5.5. Factoring the Pushout

**Motivation 5.5.1.** For a pasting scheme \( \mathcal{G} \) and a \( \mathcal{G} \)-prop \( P \) in \( M \), we think of the \( n \)th filtration stratum \([5.1.4]\)
\[ W(\mathcal{G}, J, P)|_{n+1}(\frac{\mathcal{G}}{}); \]

of the \( W \)-construction as the space of decorated graphs \((J \otimes P)[G]\), in which \( G \) has at most \( n \) ordinary internal edges, with relations parametrized by the category \( \mathcal{G}_n(\frac{\mathcal{G}}{}); \). The \( \mathcal{G} \)-prop structure on the \( W \)-construction \([3.5.6]\) is defined via graph substitution, where newly created ordinary internal edges are given length 1. For the purpose of proving the cofibrancy of the \( W \)-construction, we would like to capture the image of such a \( \mathcal{G} \)-prop structure map in the filtration. In other words, within the \((n + 1)\)st filtration stratum \( W(\mathcal{G}, J, P)|_{n+1}(\frac{\mathcal{G}}{}) \), we would like to define the object \( W(\mathcal{G}, J, P)|_{n+1}(\frac{\mathcal{G}}{}) \) that includes:

- the previous filtration stratum \( W(\mathcal{G}, J, P)|_{n}(\frac{\mathcal{G}}{}) \);
- the image of any \( \mathcal{G} \)-prop structure map \( \gamma \),

\[ W(\mathcal{G}, J, P)|_{n}(\frac{\mathcal{G}}{}) \]
\[ W(\mathcal{G}, J, P)|[G] \xrightarrow{\gamma} W(\mathcal{G}, J, P)|_{n+1}(\frac{\mathcal{G}}{}) \]

that factors through \( W(\mathcal{G}, J, P)|_{n+1}(\frac{\mathcal{G}}{}) \).
5.5. FACTORING THE PUSHOUT

This object \( W(\mathcal{G}, J, P)^n(\tilde{\tau}) \) is defined precisely in the next observation. The precise formulation of the second property above is Theorem 6.2.2 below.

**Proposition 5.5.2.** For each \( n \geq 0 \), each pair \( (\tilde{\tau}, \tilde{\sigma}) \) of \( \mathcal{C} \)-profiles, and each \( \mathcal{G} \)-prop \( P \) in \( M \) with \( \mathcal{G} \) a \( \mathcal{C} \)-colored connected unital pasting scheme, there is a commutative diagram (5.5.3)

\[
\begin{array}{cccc}
\text{colim}_{G \in \mathcal{G}_{n+1}(\tilde{\tau})} (J \otimes P)^{-}[G] & \xrightarrow{\xi} & \text{colim}_{G \in \mathcal{G}_{n+1}(\tilde{\tau})} (J \otimes P)^{\circ}[G] & \xrightarrow{\delta^*} & \text{colim}_{G \in \mathcal{G}_{n+1}(\tilde{\tau})} (J \otimes P)[G] \\
\rho \downarrow & & \rho^* \downarrow & & \downarrow \rho^* \\
W(\mathcal{G}, J, P)_n(\tilde{\tau}) & \to & W(\mathcal{G}, J, P)^\circ_n(\tilde{\tau}) & \to & W(\mathcal{G}, J, P)_{n+1}(\tilde{\tau})
\end{array}
\]

such that:

- Both squares are defined as pushouts.
- The top wedge is from Lemma 5.4.2.
- The outermost diagram is the pushout square in Theorem 5.4.7.

**Proof.** Since \( \delta^* \) factors as \( \delta^* \xi \) by Lemma 5.4.2, we can first define the left square in (5.5.3) as a pushout of \( \rho \) and \( \xi \), and then define the right square as a pushout of \( \rho^* \) and \( \delta^* \). Since two consecutive pushouts together form a pushout, the iterated pushout must agree with the pushout of \( \rho \) and \( \delta^* \) from Theorem 5.4.7 by the uniqueness of pushouts. \( \square \)

**Proposition 5.5.4.** In the context of the previous Proposition, the \( \Sigma_\mathcal{C} \)-bimodule structure on \( W(\mathcal{G}, J, P)_n \) (Lemma 5.1.12) extends to a \( \Sigma_\mathcal{C} \)-bimodule structure on \( W(\mathcal{G}, J, P)^\circ_n \) such that both maps

\[
W(\mathcal{G}, J, P)_n \longrightarrow W(\mathcal{G}, J, P)^\circ_n \longrightarrow W(\mathcal{G}, J, P)_{n+1}
\]

are maps of \( \Sigma_\mathcal{C} \)-bimodules.

**Proof.** For a graph \( G \in \mathcal{G}(\tilde{\tau}) \) and permutations \((\tau; \sigma)\) for which \((\tilde{\sigma} \tilde{\tau})\) makes sense, the graph \( \sigma G \tau \in \mathcal{G}(\tilde{\sigma} \tilde{\tau}) \) is obtained from \( G \) by reordering its profile using \((\tau; \sigma)\). In terms of graph substitution, if \( C \) is the corolla with profiles \((\tilde{\tau})\), then

\[
\sigma G \tau = (\sigma C \tau)(G),
\]

where \( \sigma C \tau \) is a permuted corolla. The graphs \( G \) and \( \sigma G \tau \) have the same sets of tunnels, so there is a natural isomorphism

\[
P^{-}[G] \cong P^{-}[\sigma G \tau].
\]

Likewise, there is a natural isomorphism of decomposition categories

\[
D(G) \cong D(\sigma G \tau)
\]

induced by the map that sends an object

\[(H_v) : G \longrightarrow K\]

in \( D(G) \) to the object

\[(H_v) : \sigma G \tau \longrightarrow \sigma K \tau\]
in $D(\sigma G \tau)$. This induces a natural isomorphism

$$J^{-}[G] \simeq J^{-}[\sigma G \tau].$$

These maps are compatible with the maps $\rho$ and $\xi$. As each entry of $W(\mathcal{G}, J, P)_n$ is a pushout of $\rho$ and $\xi$, there is an induced equivariant structure on $W(\mathcal{G}, J, P)_n$. The above maps are also compatible with $\delta^*$, so the map

$$W(\mathcal{G}, J, P)_n \longrightarrow W(\mathcal{G}, J, P)_{n+1}$$

is also equivariant.

\[\square\]

5.6. Filtering Coends

This is for Mark’s stuff about the categorical context for filtering coends.
Maps from the Boardman-Vogt Construction

We continue to assume that $\langle M, \otimes, 1 \rangle$ is a cocomplete symmetric monoidal category in which the monoidal product commutes with colimits in both variables, and $(J, \mu, 0, 1, \epsilon)$ is a commutative segment in $M$. Suppose $G = (S, G)$ is a unital pasting scheme.

In this chapter, we use the filtration in Prop. 5.1.15 to study maps out of the $W$-construction. In Section 6.1 we reinterpret a map from the $W$-construction as a compatible sequence of maps out of the filtration. In Section 6.2 and Section 6.3 we study extension of a map from one stratum to the next. These results will be used in the next chapter to show that $W(G, J, P)$ is a cofibrant $G$-prop for nice enough $P$.

6.1. Maps from the Strata

The following concept will allow us to build a map out of the $W$-construction through its strata. Recall that $|G|$ denotes the set of ordinary internal edges in a graph $G$ or its cardinality.

**Motivation 6.1.1.** For a nice enough $G$-prop $P$ in $M$, we will later show that the augmentation in Prop. 4.1.2

$$\eta : W(G, J, P) \longrightarrow P$$

is a cofibrant resolution of $P$. An object in a model category is cofibrant if the unique map from the initial object has the left lifting property with respect to acyclic fibrations. So eventually we will have to prove that maps out of the $W$-construction have a certain lifting property. We will prove this using the filtration $\{W(G, J, P)_n\}$. To make use of the filtration, first we need to understand maps from the $W$-construction as a $G$-prop in terms of the filtration strata. The following definition is a precise way to phrase this.

**Definition 6.1.2.** Suppose $P$ is a $G$-prop in $M$. Given a $G$-prop $Q$ in $M$, an $n$-map for $n \geq 0$ is a collection of maps

$$\left\{ W(G, J, P)_k \xrightarrow{\phi_k} Q : 0 \leq k \leq n \right\}$$

of $\Sigma^c$-bimodules in $M$ such that the following two conditions hold.

1. The restriction of $\phi_{k+1}$ to $W(G, J, P)_k$ is $\phi_k$ for each $0 \leq k \leq n - 1$.
2. Given a map

$$\left( H_v \right) : G \longrightarrow K \in \mathcal{G}_k(\frac{\mathcal{F}}{2})$$

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with $0 \leq k \leq n$, the diagram

\[
\begin{array}{ccccccccc}
\otimes_{v \in K} (J \otimes P)[H_v] & \xrightarrow{\pi} & (J \otimes P)[G] & \xrightarrow{\omega_G} & W(G, J, P)_{|G|}(\mathbb{Z}) \\
\otimes_{v \in K} \omega_{H_v} & & & & & & & & & \\
\otimes_{v \in K} W(G, J, P)_{|H_v|}(v) & & & & & & & & & \\
\otimes_{v \in K} \phi_{|H_v|} & & & & & & & & & \\
\otimes_{v \in K} Q(v) & \xrightarrow{=} & Q[K] & \xrightarrow{\gamma_{\mathbb{Z}}} & Q(\mathbb{Z})
\end{array}
\]

is commutative. Here the map $\pi$ (3.5.6) is part of the $G$-prop structure map of $W(G, J, P)$, and each $\omega$ (5.1.5) is the natural map to $W(G, J, P)_{|G|}(\mathbb{Z})$.

Note that

\[
\bigsqcup_{v \in K} |H_v| \leq |G|
\]

because each ordinary internal edge in $H_v$ becomes a distinct ordinary internal edge in $G$. So each map $\phi_{|H_v|}$ is defined.

**Proposition 6.1.4.** In the context of Def. [6.1.2] a map

\[
\phi : W(G, J, P) \longrightarrow Q
\]

of $G$-props is equivalent to a collection of maps $\{\phi_n\}_{n \geq 0}$ defined as the restrictions

\[
\begin{array}{cccc}
W(G, J, P)_n & \longrightarrow & W(G, J, P) & \phi \\
\phi_n & & & \\
\end{array}
\]

such that $\{\phi_k\}_{0 \leq k \leq n}$ is an $n$-map for each $n \geq 0$.

**Proof.** Suppose $\phi$ is a map of $G$-props, and

\[(H_v) : G \rightarrow K \in \mathcal{G}_{\mathbb{Z}}(\mathbb{Z})\]

for some $k \geq 0$. With the abbreviation $W = W(G, J, P)$, consider the diagram

\[
\begin{array}{ccccccccc}
\otimes_{v \in K} (J \otimes P)[H_v] & \xrightarrow{\pi} & (J \otimes P)[G] & \xrightarrow{\omega_G} & W(G, J, P)_{|G|}(\mathbb{Z}) \\
\otimes_{v \in K} \omega_{H_v} & & & & & & & & & \\
\otimes_{v \in K} W[H_v](v) & \xrightarrow{(\ast)} & W[K] & \xrightarrow{\gamma_{\mathbb{Z}}} & W(\mathbb{Z}) & \xrightarrow{\phi_{|G|}} & W[G](\mathbb{Z}) \\
\otimes_{v \in K} \phi_{|H_v|} & & & & & & & & & \\
\otimes_{v \in K} Q(v) & \xrightarrow{=} & Q[K] & \xrightarrow{\gamma_{\mathbb{Z}}} & Q(\mathbb{Z}) & \xrightarrow{=} & Q(\mathbb{Z})
\end{array}
\]

in which the outermost diagram is (6.1.3) and $\gamma_W = \gamma^{W(G, J, P)}$. The sub-diagram $(\ast)$ is commutative by the definition of the $G$-prop structure maps of $W(G, J, P)$ (3.5.5). The sub-diagram $(\ast \ast)$ is commutative because $\phi$ is a map of $G$-props. All other sub-diagrams are commutative by definition. This proves that $\{\phi_k\}_{0 \leq k \leq n}$ is an $n$-map for each $n \geq 0$. 


Conversely, suppose given a collection of maps \( \{ \phi_n \}_{n \geq 0} \) such that \( \{ \phi_k \}_{0 \leq k \leq n} \) is an \( n \)-map for each \( n \geq 0 \). By Prop. 5.1.15 these maps determine a unique map

\[
\phi : W(\mathcal{G}, J, P) \rightarrow Q
\]

at each entry. It remains to prove that \( \phi \) preserves the \( \mathcal{G} \)-prop structure maps \( \gamma_K \) for \( K \in \mathcal{G}(2) \). Reusing the diagram (6.1.5), we must show that the sub-diagram (**) is commutative. Since the \( \mathcal{G} \)-prop structure map \( \gamma_W^K \) is defined by the sub-diagram (**) for \( H_v \in \mathcal{G}(v) \) for \( v \in K \) and \( G = K(H_v) \), it is enough to show that the combined sub-diagram of (**) is commutative. Since all other sub-diagrams in (6.1.5) are commutative, we conclude the proof by observing that the outermost diagram in (6.1.5) is the diagram (6.1.3), which is commutative by assumption. \( \square \)

**Example 6.1.6.** Let us illustrate the concept of an \( n \)-map with the \( \mathcal{C} \)-colored pasting scheme \( \mathcal{G} = \mathcal{U} \text{Lin} \) of unital linear graphs, \( M = \text{Top} \), and \( J \) the unit interval \(([0, 1], \ast) \) equipped with either operations

\[
a \ast b = \max\{a, b\} \quad \text{or} \quad a \ast b = 1 - (1 - a)(1 - b).
\]

A \( \mathcal{C} \)-colored linear graph with \( n \geq 1 \) vertices and \( n - 1 \) internal edges looks like

\[
L(c_0, \ldots, c_n) = c_0 \rightarrow 1 \rightarrow c_1 \rightarrow 2 \rightarrow \cdots \rightarrow c_{n - 1} \rightarrow n \rightarrow c_n
\]

with each \( c_i \in \mathcal{C} \).

Suppose \( P \) is a \( \mathcal{U} \text{Lin} \)-prop in \( \mathcal{M} \), i.e., a \( \text{Top} \)-enriched category with object set \( \mathcal{C} \). We will abbreviate \( W(\mathcal{G}, J, P) \) to \( WP \). For \( c, d \in \mathcal{C} \), the \( n \)th filtration stratum

\[
WP_n(c, d) = \int_{L \in \mathcal{U} \text{Lin}_{\mathcal{C}}} J(L) \times P(L)
\]

is a quotient of the space of sequences

\[
f_m \stackrel{t_{m - 1}}{\circ} \cdots \stackrel{t_2}{\circ} \stackrel{t_1}{\circ} f_1 = \left\{ \begin{array}{ll}
\cdots f_{i+1} \circ \text{Id}_{c_i} \circ t_i \circ f_{i-1} & \text{if } i = 1; \\
\cdots f_{i+1} \circ t_i \circ f_{i-1} & \text{if } 1 < i < m; \\
f_m \circ t_{m - 2} \circ \cdots \circ t_1 \circ f_1 & \text{if } i = m.
\end{array} \right.
\]

with \( m \leq n + 1 \), each \( t_j \in J = [0, 1] \), \( f_i \in P(c_{i-1}, c_i) \) for \( 1 \leq i \leq m \) and some \( c_j \) for \( 1 \leq j \leq m - 1 \), \( c_0 = c \), and \( c_m = d \). So the \( f_i \)’s are composable maps in \( P \) such that the domain of \( f_1 \) is \( c \) and that the codomain of \( f_m \) is \( d \). The empty sequence, corresponding to \( m = 0 \), is allowed if and only if \( d = c \).

These sequences are subject to two identifications:

**Units:** If \( f_i \in P(c_{i-1}, c_i) \) is the identity map of \( c_i \), then

\[
\cdots f_{i+1} \circ \text{Id}_{c_i} \circ t_i \circ f_{i-1} \cdots = \left\{ \begin{array}{ll}
f_m \circ t_{m - 1} \circ \cdots \circ t_2 \circ f_2 & \text{if } i = 1; \\
\cdots f_{i+1} \circ t_i \circ f_{i-1} \cdots & \text{if } 1 < i < m; \\
f_m \circ t_{m - 2} \circ \cdots \circ t_1 \circ f_1 & \text{if } i = m.
\end{array} \right.
\]

In other words, if \( f_i = \text{Id}_{c_i} \) is neither the left-most nor the right-most entry, then we may delete it and compose the two adjacent \( t_j \)’s in \( J \). For example, we have

\[
f_4 \circ t_4 \circ \text{Id}_{c_3} \circ t_2 \circ f_2 \circ t_1 \circ f_1 = f_4 \circ t_4 \circ t_2 \circ t_1 \circ f_1.
\]
If \( f_i = \text{Id}_c \) is the left-most or the right-most entry, then we may delete it along with the adjacent \( t \in J \).

**Composition:** If \( t_i = 0 \), then
\[
\ldots f_{i+1} \circ f_i \ldots = \ldots (f_{i+1} f_i) \ldots.
\]
In other words, if \( t_i = 0 \), then we may delete it and compose the two adjacent \( f_j \)'s in \( P \). For example, we have
\[
f_4 \circ f_3 \circ f_2 \circ f_1 = f_4 \circ f_3 f_2 \circ f_1.
\]

Suppose \( Q \) is another \( \text{Top} \)-enriched category with object set \( \mathcal{C} \). An \( n \)-map \( \{ \phi_k \}_{k=0}^n \) consists of compatible maps
\[
\begin{align*}
\xymatrix{ WP_0(c) & WP_1(c) & \cdots & WP_n(c) \\
\phi_0 & \phi_1 & \cdots & \phi_n \\
\end{align*}
\]
for \( c, d \in \mathcal{C} \) that make the diagram (6.1.3) commutative. For example, suppose we have two sequences
\[
f_3 \circ f_2 \circ f_1 \in WP_2(c) \quad \text{and} \quad g_3 \circ g_2 \circ g_1 \in WP_3(c).
\]
Then an \( n \)-map with \( n \geq 6 \) satisfies the equality
\[
\phi_0 \left( g_3 \circ g_2 \circ g_1 \circ f_1 \right) = \phi_3 \left( \phi_2 \left( g_3 \circ g_2 \circ g_1 \circ f_1 \right) \right)
\]
in \( Q(c) \). The sequence on the left \( g_3 \circ \ldots \circ f_1 \) comes from the graph substitution
\[
\xymatrix{ f_1 & f_2 & f_3 \\
& c & b & d \\
}
\]
which gives the \( b \)-colored internal edge length 1, corresponding to \( 0 \).

**6.2. Extending Maps Up Half of a Stratum**

**Motivation 6.2.1.** For a pasting scheme \( G \) and a \( G \)-prop \( P \) in \( M \), we are in the process of showing that the natural map between two consecutive filtration strata
\[
W(G, J, P)_n \longrightarrow W(G, J, P)_{n+1}
\]
has a certain lifting property. Using the factorization
\[
W(G, J, P)_n \longrightarrow W(G, J, P)^\circ_n \longrightarrow W(G, J, P)_{n+1}
\]
in Prop. 5.5.4, the first step is to extend an \( n \)-map to the object \( W(G, J, P)^\circ_n \). Recall that we think of the object \( W(G, J, P)^\circ_n \) as the subspace of the filtration stratum \( W(G, J, P)_{n+1} \) containing \( W(G, J, P)_n \) and all the decorated graphs with \( |G| = n + 1 \) that are images of any \( G \)-prop structure map of the \( W \)-construction. These structure maps of \( W(G, J, P) \) (3.5.6) give newly created ordinary internal edges length 1.
Suppose we are given an $n$-map

$$\{ \phi_k : W(\mathcal{G}, J, P)_k \rightarrow Q \}_{k=0}^n$$

a $\mathcal{G}$-prop structure map

$$W(\mathcal{G}, J, P)[K] = \bigotimes_{v \in K} W(\mathcal{G}, J, P)(v) \xrightarrow{\gamma_K} W(\mathcal{G}, J, P)(\emptyset)$$

with $|K| \geq 1$, and $H_v \in \mathcal{G}(v)$ with $|H_v| \leq n$ for each vertex $v \in K$ such that $|K(H_v)| = n + 1$. In order for the desired extension to $W(\mathcal{G}, J, P)_n^\circ$ to be compatible with $\mathcal{G}$-prop structure maps, we must send

$$\gamma_K \left( (J \otimes P)[H_v] \right)_{v \in K} \in W(\mathcal{G}, J, P)_n^\circ(\emptyset)$$

In other words, for each vertex $v \in K$ send $(J \otimes P)[H_v]$ to $Q(v)$ using the given maps $\phi_k$, and then compose them in $Q$ using the $\mathcal{G}$-prop structure map $\gamma_K^Q$. The next observation is the precise manifestation of this heuristic discussion.

**Theorem 6.2.2.** Suppose $P$ and $Q$ are $\mathcal{G}$-props in $M$ with $\mathcal{G}$ a $\mathcal{C}$-colored connected unital pasting scheme, and

$$\{ \phi_k : W(\mathcal{G}, J, P)_k \rightarrow Q \}_{0 \leq k \leq n}$$

is an $n$-map for some $n \geq 0$. Suppose

(6.2.3)

is a commutative solid-arrow diagram of $\mathcal{G}$-$\mathcal{E}$-bimodules in $M$ such that $\psi$ and $\chi$ are maps of $\mathcal{G}$-props and that $W(\mathcal{G}, J, P)_n^\circ$ is from Prop. 5.5.2. Then there exists a canonical map

$$\phi_n^\circ : W(\mathcal{G}, J, P)_n^\circ \rightarrow Q$$

of $\mathcal{G}$-$\mathcal{E}$-bimodules in $M$ that makes the diagram (6.2.3) commutative.

**Proof.** To define the map $\phi_n^\circ$, since it must extend $\phi_n$, using the left pushout square in (5.5.3), we now define a map

$$(J \otimes P)^\circ[G] \rightarrow Q \quad \text{for} \quad G \in \mathcal{G}_{\geq n+1}(\emptyset).$$

By the pushout definitions of $(J \otimes P)^-[G]$ and $(J \otimes P)^+[G]$ (5.2.10), the restriction of $\phi_n^\circ$ to $J[G] \otimes P^-[G]$ (resp., $J^-[G] \otimes P[G]$) must be the same as the restriction of $\phi_n \rho$, where $\rho$ is the map in Lemma 5.3.3. It remains to define the restriction of $\phi_n^\circ$ to

$$\left( \colim_{G=K(H_v) \otimes \mathcal{G}(G)} [K] \otimes \bigotimes_{v \in K} J[H_v] \right) \otimes P[G]$$

in which the colimit on the left is from (5.2.9).
For each object 

\[(H_v): G \longrightarrow K\]

in \(\mathcal{D}^\circ(G)\) (so \(|K| \geq 1\) and \(\bigcup_{v \in K} |H_v| < |G|\)), define the restriction of \(\phi_n^o\) as the composite \(\phi_n^o\)

\[
\begin{align*}
\left( \right. & \left. \prod_{v \in K} \left[ I[K] \otimes \bigotimes_{v \in K} J[H_v] \right] \otimes P[G] \right) \xrightarrow{\phi_n^o} \bigotimes_{v \in K} Q(v) \\
\xrightarrow{z} & \bigotimes_{v \in K} (J \otimes P)[H_v] \otimes W[H_v](v) \otimes P[G] \xrightarrow{\phi_n^o} \bigotimes_{v \in K} Q(v) = Q[K]
\end{align*}
\]

in which \(W = W(\mathcal{G}, J, P)\). In Lemma 6.2.7 below we will show that the above maps yield a well defined map

\[
\left( \right. & \left. \prod_{v \in K} \left[ I[K] \otimes \bigotimes_{v \in K} J[H_v] \right] \otimes P[G] \right) \xrightarrow{\phi_n^o} \bigotimes_{v \in K} Q(v)
\]

Then in Lemma 6.2.8 we will show that the above map yields a well defined map

\[
W(\mathcal{G}, J, P)^o_n \xrightarrow{\phi_n^o} Q
\]

of \(\Sigma\varepsilon\)-bimodules in \(\mathcal{M}\). Finally, in Lemma 6.2.10 below we will show that this map \(\phi_n^o\) makes the diagram (6.2.3) commutative.

**Lemma 6.2.7.** The map

\[
\left( \right. & \left. \prod_{v \in K} \left[ I[K] \otimes \bigotimes_{v \in K} J[H_v] \right] \otimes P[G] \right) \xrightarrow{\phi_n^o} \bigotimes_{v \in K} Q(v)
\]

in (6.2.5) is well defined.

**Proof.** Since \(\phi_n^o\) is defined at each object of \(\mathcal{D}^\circ(G)\) as in (6.2.4), it is enough to show that, for each map

\[
\left( \right. & \left. (H_v): G \longrightarrow K \right) \longrightarrow \left( \right. & \left. (H'_v): G \longrightarrow K' \right)
\]

in \(\mathcal{D}^\circ(G)\) as in (5.2.14), the outermost diagram in

\[
\begin{align*}
\bigotimes_{v \in K} (J \otimes P)[H_v] & \xrightarrow{\bigotimes_{v \in K} \phi_n^o} \bigotimes_{v \in K} Q(v) \\
\bigotimes_{u \in K''/\{H_u\}} 1 & \xrightarrow{\bigotimes_{u \in K''} \omega_{H_u}} \bigotimes_{v \in K} W[H_v](v) \otimes P[G] \xrightarrow{\bigotimes_{v \in K} \phi_n^o} \bigotimes_{v \in K} Q(v) \\
\bigotimes_{u \in K''} \omega_{H_u} & \xrightarrow{\bigotimes_{u \in K''} \gamma_n^D} \bigotimes_{u \in K'} W[H'_u](u) \otimes P[G] \xrightarrow{\bigotimes_{u \in K'} \phi_n^o} \bigotimes_{u \in K'} Q(u) = Q[K'] \\
\bigotimes_{u \in K''} \omega_{H_u} & \xrightarrow{\bigotimes_{u \in K''} \gamma_n^D} \bigotimes_{u \in K'} W[H'_u](u) \otimes P[G] \xrightarrow{\bigotimes_{u \in K'} \phi_n^o} \bigotimes_{u \in K'} Q(u) = Q[K'] \\
\bigotimes_{u \in K''} \omega_{H_u} & \xrightarrow{\bigotimes_{u \in K''} \gamma_n^D} \bigotimes_{u \in K'} W[H'_u](u) \otimes P[G] \xrightarrow{\bigotimes_{u \in K'} \phi_n^o} \bigotimes_{u \in K'} Q(u) = Q[K']
\end{align*}
\]

is commutative, in which \(W = W(\mathcal{G}, J, P)\). The upper left vertical map and the slanted map are defined because of the decompositions

\[
K = K'(D_u) \quad \text{and} \quad H'_u = D_u(H_{uv})
\]
in (6.2.13). The lower right triangle is commutative by the associativity of the \( G \)-prop structure map \( \gamma^Q \). Since each \( |H_u'| < |G| = n + 1 \), it follows that each \( |D_u| < n + 1 \). Moreover, we have that
\[
\bigotimes_{v \in K} \bigotimes_{u \in K'} \bigotimes_{v \in D_u} \]
since \( K = K'(D_u) \). The trapezoidal sub-diagram is commutative because it is the tensor product over \( u \in K' \) of diagrams, each of which is commutative by (6.1.3) since
\[
(H_{uv}) : H_u' \longrightarrow D_u
\]
is a map in \( G_n(u) \) for each \( u \in K' \). □

**Lemma 6.2.8.** The map in Lemma 6.2.7 yields a well defined map

\[
W(G, J, P)_n \longrightarrow \phi_n^0 Q
\]
of \( \Sigma \)-bimodules in \( M \).

**Proof.** It suffices to show that \( \phi_n^0 \) defined in (6.2.4) is compatible with the pushout square in (6.2.9) tensored with \( P[G] \). So suppose
\[
(H_v) : G \longrightarrow K \in \mathcal{G}_{ord}(2)
\]
with
\[
|G| = n + 1, \quad |K| \geq 1, \quad \text{and} \quad |H_x| \geq 1
\]
for some vertex \( x \in K \). For vertices \( v \in K \), we define
\[
H'_v = \begin{cases} 
H_v & \text{if } v \neq x, \\
C_x & \text{if } v = x
\end{cases}
\]
with \( C_x \) the corolla with the same profile as \( x \). Since \( |H_x| \geq 1 \), it follows that \( |K(H'_v)| < |G| \). We must show the outermost diagram in (6.2.9)
is commutative. Here $0, 1: \mathbb{I} \rightarrow J$ are part of the commutative segment $J$, $W = W(G, J, P)$, and

$$Y = J[C_x] \otimes P[H] \otimes \bigotimes_{v \in K \setminus \{x\}} (J \otimes P)[H_v].$$

Undecorated maps are from the filtration on $W$ (Prop. 5.1.15). In the diagram (6.2.9):

- The sub-diagram 1 is commutative because $P$ is a $G$-prop.
- The sub-diagram 2 is commutative by the definition of $\rho$ restricted to $J [-G] \otimes P[G]$ (5.3.8).
- The sub-diagram 3 is commutative by the definition of $W_n(\xi)$ as a coend over $G_n(\xi)$ and the fact that

$$\phi_n: W(G, J, P)^{\circ}_n \longrightarrow Q$$

in Lemma 6.2.8 makes the diagram (6.2.3) commutative. 

**PROOF.** The map $\phi_n$ is constructed as an extension of $\phi_n$, so it suffices to show that the diagram

$$\begin{array}{ccc}
W_n(\xi) & \xrightarrow{\phi_n} & Q(\xi) \\
\downarrow \psi & & \downarrow \chi \\
W(\xi) & \xrightarrow{\psi} & T(\xi)
\end{array}$$

is commutative for each pair $(\xi)$ of $C$-profiles, in which $W = W(G, J, P)$. By the pushout definition of $W(G, J, P)^{\circ}_n$, it suffices to show that, for each object

$$(H_v): G \longrightarrow K \in D^{\circ}(G),$$
the outermost diagram in (6.2.11)

\[ (J \otimes P)[H_v] \xrightarrow{\otimes (J \otimes P)[H_v]} \otimes_{v \in K} W(H(v)) \xrightarrow{\otimes_{v \in K} \phi[H_v]} \otimes_{v \in K} Q(v) = Q[K] \]

is commutative.

In the diagram (6.2.11):

- The sub-diagram 1 is commutative by the definition of \( \beta_v \) (Def. 5.2.9).
- The combined sub-diagram of 1 and 2 is commutative by the definition of \( \gamma^W \) (Def. 3.5.3).
- The sub-diagram 3 is the tensor product over \( v \in K \) of the diagrams

\[ W[H(v)] \xrightarrow{\phi[H_v]} Q(v) \]

\[ W(v) \xrightarrow{\psi} T(v), \]

each of which is assumed commutative (6.2.3).

- The sub-diagram 4 (resp., 5) is commutative because \( \psi \) (resp., \( \chi \)) is a map of \( G \)-props.
- The remaining unnumbered sub-diagram is commutative by the filtration on \( W \) (Prop. 5.1.15).

So the diagram (6.2.11) is commutative, and hence so is (6.2.3). □

The proof of Theorem 6.2.2 is now complete.

Example 6.2.12. As an illustration of the map \( \phi^n \) in (6.2.4), consider the \( C \)-colored pasting scheme \( Gr^{\gamma}_c \) of connected wheeled graphs and a \( G \)-prop \( P \) (i.e., a \( C \)-colored wheeled properad) in \( M \). We will write \( WP \) for \( W(G, J, P) \). The \( n \)th filtration stratum is defined as the coend

\[ WP_n(\gamma) = \int_{G \in Gr^{\gamma}_c} J[G] \otimes P[G]. \]

Suppose we are given a 1-map to some \( G \)-prop \( Q \). Consider the object
in $D^\circ(G)$. The corresponding map $\phi_2^\circ$ is the following composite:

$$1[K] \otimes J[H] \otimes P[G] = 1_c \otimes J_a \otimes P(u) \otimes P(v) \xrightarrow{\phi_2^\circ} Q(d') \xrightarrow{\gamma_Q^K} Q[K]$$

Note that since $K$ is a contracted corolla, the structure map $\gamma_Q^K$ is a contraction.

6.3. Extending Maps Up One Stratum

**Motivation 6.3.1.** Suppose given a pasting scheme $G$, a $G$-prop $P$ in $M$, and an $n$-map

$$\left\{ \phi_k : W(G, J, P)_k \longrightarrow Q \right\}_{k=0}^n$$

to some $G$-prop $Q$. We are in the process of constructing an extension of this $n$-map to an $(n+1)$-map using the horizontal factorization

$$W(G, J, P)_n \xrightarrow{\phi_n} W(G, J, P)^n \xrightarrow{\phi_n^\circ} W(G, J, P)_{n+1} \xrightarrow{\phi_{n+1}^\circ} Q$$

in Prop. 5.3.4. The first step—namely, extending the given $n$-map to $\phi_n^\circ$—was achieved in Theorem 6.2.2. The remaining step is to extend $\phi_n^\circ$ to a map $\phi_{n+1}$ such that the collection $\left\{ \phi_k \right\}_{k=0}^{n+1}$ is an $(n+1)$-map.

The next observation says that, as long as $\phi_{n+1}$ is an extension of $\phi_n^\circ$ that respects the equivariant structure, then it is guaranteed to yield an $(n+1)$-map. The existence of such an extension $\phi_{n+1}$ cannot in general be inferred from just the combinatorics of graphs and the algebraic properties of generalized props. We will eventually use model categorical methods to establish the existence of such an extension $\phi_{n+1}$.
Theorem 6.3.2. Under the same hypotheses as in Theorem 6.2.2, suppose $\phi_n$ is the map constructed there. If there exists a dotted filler $\phi_{n+1}$ of $\Sigma_{\mathcal{C}}$-bimodules in $\mathcal{M}$ that makes the whole diagram commutative, then $\{\phi_k\}_{0 \leq k \leq n+1}$ is an $(n+1)$-map.

Proof. We must show that the diagram (6.1.3) is commutative when the index is $n+1$. Suppose given a map $\left(\mathcal{H}_v\right) : G \longrightarrow K \in \mathcal{G}_{n+1}(\mathcal{J}, \mathcal{P})$ of $\Sigma_{\mathcal{C}}$-bimodules, we may assume that $K \geq 1$. By the connectivity of $G$, no $\mathcal{H}_v$ can be an exceptional loop. We may further assume that no $\mathcal{H}_v$ is an exceptional edge because substituting in an exceptional edge reduces the number of ordinary internal edges, and we already assumed that $\{\phi_k\}_{0 \leq k \leq n}$ is an $n$-map. Therefore, now we have that $\left(\mathcal{H}_v\right) : G \longrightarrow K$ is an object in $\mathcal{D}^*(G)$.

The diagram (6.1.3) for $n+1$ is the outermost diagram below, in which $W = W(\mathcal{G}, J, P)$.

\begin{equation}
\begin{array}{c}
\otimes_{v \in K} (J \otimes P)[\mathcal{H}_v] \\
\downarrow \otimes_{v \in K} \omega_{\mathcal{H}_v} \\
\otimes_{v \in K} W_{\mathcal{H}_v}(v)
\end{array}
\begin{array}{c}
\pi \\
J^*G \otimes P[G] \\
\otimes_{v \in K} \phi_{\mathcal{H}_v}
\end{array}
\begin{array}{c}
( J \otimes P)[G] \\
\beta_G \\
\otimes_{v \in K} \phi_{\mathcal{H}_v}
\end{array}
\begin{array}{c}
\omega_G \\
\beta^* \\
\phi_{n+1}
\end{array}
\begin{array}{c}
W_{n+1}(\mathcal{J}) \\
W_\mathcal{G}^* \mathcal{J} \\
W_\mathcal{G}^* \mathcal{J} \\
\omega \mathcal{K}
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longrightarrow \\
\longrightarrow
\end{array}
\begin{array}{c}
W(\mathcal{G}, J, P) \\
W(\mathcal{G}, J, P) \\
W(\mathcal{G}, J, P) \\
Q(\mathcal{J})
\end{array}
\begin{array}{c}
\longrightarrow \\
\longrightarrow \\
\longrightarrow \\
\longrightarrow
\end{array}
\begin{array}{c}
Q(\mathcal{J}) \\
Q(\mathcal{J}) \\
Q(\mathcal{J}) \\
Q(\mathcal{J})
\end{array}
\end{equation}

In the diagram (6.3.4):
- The unnumbered triangle on the right is commutative by assumption.
- The sub-diagram $\square$ is commutative by the definitions of $\pi$ (3.5.6) and of $\beta_G^*$ (5.2.9).
- The sub-diagram $\square$ is commutative by the right pushout square in (5.5.3).
- The sub-diagram $\square$ is commutative by the definition of $\phi_{n+1}$ (6.2.4).

Therefore, the diagram (6.1.3) for $n+1$ is commutative.

Example 6.3.5. Let us explain why in Theorem 6.3.2 we did not simply prove the existence of the map $\phi_{n+1}$. In Motivation 6.3.1 we briefly mentioned that the existence of $\phi_{n+1}$ cannot be established without some model categorical assumptions. For instance, consider the setting of Example 6.1.6 with $\mathcal{G} = \text{ULin}$ the $\mathcal{C}$-colored pasting scheme of unital linear graphs, $\mathcal{M} = \text{Top}$, $J = [0,1]$, and $P$ a $\text{Top}$-enriched...
category with object set \( \mathcal{C} \). The \( n \)th filtration stratum \( WP_n(c) \) is the space of sequences

\[
f_m \circ t_{m-1} \circ \cdots \circ t_2 \circ t_1
\]
with \( m \leq n + 1 \), the \( f_i \)'s composable maps in \( P \) starting at \( c \) and ending at \( d \), and each \( t_j \in [0, 1] \). There are two identifications among these sequences, one involving the identity maps in \( P \) and the other involving composition in \( P \).

The object \( WP_n^c(d) \) is the subspace of \( WP_{n+1}(c) \) consisting of sequences as above such that either

1. \( m \leq n + 1 \) or
2. \( m = n + 2 \) and at least one \( t_j = 1 \) for some \( 1 \leq j \leq n + 1 \).

However, in \( WP_{n+1}(c) \) there are sequences

\[
f_{n+2} \circ t_{n+1} \circ \cdots \circ t_2 \circ t_1
\]
in which \( t_j \neq 0 \) for any \( 1 \leq j \leq n + 1 \). Such a sequence is not in the image of any non-identity ULin-prop structure map. So even if we have a map

\[
WP_n^c(c) \longrightarrow Q(c),
\]
we still would not know where to send these sequences.
CHAPTER 7

Resolutions of Generalized Props

The main objective of this chapter is to show that, for a nice enough $\mathcal{G}$-prop $\mathcal{P}$ in $\mathcal{M}$, the $W$-construction of $\mathcal{P}$ is a cofibrant resolution of $\mathcal{P}$ as a $\mathcal{G}$-prop. The necessary homotopical setting is explained in Section 7.1. In Section 7.2 we observe that the augmentation

$$\eta : W(\mathcal{G}, J, \mathcal{P}) \longrightarrow \mathcal{P}$$

is a weak equivalence for nice enough $\mathcal{P}$. In Section 7.3 we show that the augmentation is furthermore a cofibrant resolution for nice enough $\mathcal{P}$. In Section 7.4 and Section 7.5 we show that the $W$-construction behaves nicely in a homotopical sense with respect to a change of commutative intervals and generalized props.

7.1. Homotopical Setting

Here we briefly recall some definitions regarding model categories. The reader is referred to the references [Fre09, Hir03, Hov99, MP12, SS00] for more details.

7.1.1. Model Categories. The definition of a model category below is from [MP12] Chapter 14. Suppose $f : A \longrightarrow B$ and $g : C \longrightarrow D$ are morphisms in a category $\mathcal{C}$. We write $f \otimes g$ if for each solid-arrow commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow & & \downarrow g \\
B & \xrightarrow{g} & D
\end{array}
$$

in $\mathcal{C}$, a dotted arrow exists that makes the entire diagram commutative. For a class $\mathcal{A}$ of morphisms in $\mathcal{C}$, define the classes of morphisms

$$\mathcal{A}^\otimes = \{ g \in \mathcal{C} | \exists a \in \mathcal{A} \}.$$ 

A pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in $\mathcal{C}$ factors $\mathcal{C}$ if each morphism $h : A \longrightarrow C$ in $\mathcal{C}$ has a factorization

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow & & \downarrow h & & \downarrow h
\end{array}
$$

such that $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

A weak factorization system in a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in $\mathcal{C}$ such that $(\mathcal{L}, \mathcal{R})$ factors $\mathcal{C}$ and that

$$\mathcal{L} = \mathcal{A}^\otimes \quad \text{and} \quad \mathcal{R} = \mathcal{A}.$$
A **model category** is a complete and cocomplete category $\mathcal{M}$ equipped with three classes of morphisms $(\mathcal{W}, \mathcal{C}, \mathcal{F})$, called weak equivalences, cofibrations, and fibrations, such that:

1. $\mathcal{W}$ has the 2-out-of-3 property.
2. $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems.

Some frequently used model categories include:

- (pointed or unpointed) simplicial sets $\mathbf{SSet}$; $\text{[Quil67]}$;
- compactly generated topological spaces $\mathbf{Top}$ $\text{[Hov99]}$;
- (bounded or unbounded) chain complexes $\text{Ch}_k$ or simplicial modules $\text{SMod}_k$ over a field $k$ of characteristic 0 $\text{[Hov99]}$;
- symmetric spectra $\mathbf{Sp}^+$ with the positive (flat) stable model structure $\text{[Shi04]}$;
- the category $\mathbf{Cat}$ of small categories with the folk model structure $\text{[Rez]}$.

In a model category, an **acyclic (co)fibration** is a map that is both a (co)fibration and a weak equivalence. An object $Z$ is **fibrant** if the unique map to the terminal object is a fibration. An object $Y$ is **cofibrant** if the unique map from the initial object is a cofibration. A **cofibrant resolution** of an object $X$ is a weak equivalence $Y \longrightarrow X$ such that $Y$ is cofibrant. In this case, we also say that $Y$ is a cofibrant resolution of $X$.

In a model category, a cofibrant resolution of an object $X$ is guaranteed to exist by the cofibration-acyclic fibration factorization axiom applied to the map $\emptyset \longrightarrow X$. However, in practice it may be difficult to describe explicitly a cofibrant resolution of a given object. This is particularly the case for model categories in which objects have multiplicative structures, such as generalized props, cyclic operads, and modular operads. This difficulty of describing explicitly cofibrant resolutions is precisely why our Boardman-Vogt construction is a valuable tool.

Suppose

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{L} & \mathcal{N} \\
R & \downarrow & \\
\emptyset & \longrightarrow & X
\end{array}
$$

is an adjunction between model categories with left adjoint $L$. Then $(L, R)$ is called a **Quillen adjunction** or a **Quillen pair** if $L$ preserves cofibrations and acyclic cofibrations. A Quillen adjunction $(L, R)$ is called a **Quillen equivalence** if for each map $f : LX \longrightarrow Y$ with $X \in \mathcal{M}$ cofibrant and $Y \in \mathcal{N}$ fibrant, $f$ is a weak equivalence in $\mathcal{N}$ if and only if its adjoint $X \longrightarrow RY$ is a weak equivalence in $\mathcal{M}$.

### 7.1.2. Cofibrantly Generated Model Categories

A model category $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ is **cofibrantly generated** if it is equipped with two sets $\mathcal{I}$ and $\mathcal{J}$ of morphisms that permit the small object argument and such that

$$
\mathcal{F} = \mathcal{J}^\circ \quad \text{and} \quad \mathcal{F} \cap \mathcal{W} = \mathcal{I}^\circ.
$$

The maps in $\mathcal{I}$ and $\mathcal{J}$ are called **generating cofibrations** and **generating acyclic cofibrations**, respectively. The examples above are all cofibrantly generated model categories.

For a cofibrantly generated model category $\mathcal{M}$ and a small category $\mathcal{D}$, the category $\mathcal{M}^\mathcal{D}$ of $\mathcal{D}$-diagrams in $\mathcal{M}$ inherits from $\mathcal{M}$ a cofibrantly generated model
category structure \cite{Hir03} (11.6.1) with fibrations and weak equivalences defined entrywise in $M$. For example, the categories $M^{\Sigma_\ast}$ of pointed $\Sigma_\ast$-bimodules and $M^G$ of round bimodules in $M$ are both under-categories of the diagram category $M^S$ (Remark 4.2.7). So they both inherit model category structures from $M$. A cofibration (resp., acyclic cofibration) in $M^D$ is called a $D$-cofibration (resp., $D$-acyclic cofibration).

### 7.1.3. Monoidal Model Categories and Admissibility.

A monoidal model category is a symmetric monoidal closed category $M$ that is also a model category such that the following pushout product axiom is satisfied:

Given cofibrations $f : A \rightarrow B$ and $g : C \rightarrow D$, the pushout product $f \Box g$ in the diagram

\[
\begin{align*}
A \otimes C & \xrightarrow{\text{Id}_A \otimes g} A \otimes D \\
B \otimes C & \xrightarrow{\text{Id}_B \otimes f} Z
\end{align*}
\]

is a cofibration, which is furthermore a weak equivalence if either $f$ or $g$ is also a weak equivalence. Here

$$Z = B \otimes C \coprod_{A \otimes C} A \otimes D$$

is the object of the pushout square.

The examples above are all monoidal model categories. This definition of a monoidal model category is from \cite{SS00} Def. 3.1. Note that, in \cite{Hov99} Def. 4.2.6, a monoidal model category has an extra condition about the monoidal unit. We do not need this extra condition in this work.

A colored operad $O$ in a monoidal model category $M$ is said to be admissible if the category of $O$-algebras is a model category with weak equivalences and fibrations defined entrywise in $M$. For each $\mathcal{C}$-colored pasting scheme $G = (S, G)$, the category of $G$-prods in a cocomplete symmetric monoidal category is the category of algebras over some colored operad by \cite{YJ15} Theorem 14.1.

**Assumption 7.1.1.** We will assume that $M$ is a cofibrantly generated monoidal model category in which the colored operad for the $\mathcal{C}$-colored connected unital pasting scheme under discussion is admissible.

In other words, if we are discussing $G$-prods in a homotopical setting, then we assume that the category of $G$-prods in $M$ admits a model structure with weak equivalences and fibrations defined entrywise in $M$.

For example, the admissibility of all colored operads, not just the ones for $G$-prods, in simplicial sets, chain complexes of $k$-modules over a field $k$ of characteristic zero, and symmetric spectra are proved in \cite{WY17} (Section 8). The cases of simplicial modules and small categories can be proved like the case of simplicial sets. The case for compactly generated topological spaces follows from \cite{BB17} Theorem 2.11.
7.2. Weak Equivalence from the Construction

Recall the following concept from [BM06].

**Definition 7.2.1.** A *commutative interval* is a commutative segment \((J, \mu, 0, 1, \epsilon)\) such that:

1. the map \((0, 1) : \mathbb{I} \sqcup \mathbb{I} \rightarrow J\) is a cofibration;
2. the counit \(\epsilon : J \rightarrow \mathbb{I}\) is a weak equivalence.

**Example 7.2.2.** The commutative segments in Example 3.2.2 are all commutative intervals when the categories \(\text{Top}, \text{SSet}, \text{Ch}(k), \text{and} \text{SMod}(k)\) are equipped with their respective standard model structures [Quil67] and when \(\text{Cat}\) is equipped with the folk model structure [Rez].

**Motivation 7.2.3.** To establish that the \(W\)-construction of a nice enough \(G\)-prop \(P\) in \(M\) is a cofibrant resolution of \(P\), we will prove that the natural map between two consecutive filtration strata

\[
W(G, J, P)_n \rightarrow W(G, J, P)_{n+1}
\]

has nice homotopical properties. Using the pushouts in Prop. 5.5.2 we will need to know that the maps

\[
\begin{align*}
\colim_{G \in \mathcal{G}[G]^\circ} (J \otimes P)^\circ [G] & \rightarrow \colim_{G \in \mathcal{G}[G]^\circ} \colim_{G \in \mathcal{G}[G]} (J \otimes P)[G] \\
\colim_{G \in \mathcal{G}[G]^\circ} (J \otimes P)^{-} [G] & \rightarrow \colim_{G \in \mathcal{G}[G]^\circ} \colim_{G \in \mathcal{G}[G]} (J \otimes P)[G]
\end{align*}
\]

have nice homotopical properties for each maximal connected sub-groupoid \([G]_w\) in \(\mathcal{G}_{\geq n+1}(\mathcal{G})\). The maps

\[
\delta_G = \alpha_G \sqcup \beta_G^G \quad \text{and} \quad \delta_G^G = \alpha_G \sqcup \beta_G^G
\]

in (5.2.10) are defined as pushout corner maps. Therefore, we will need to make sure that the maps \(\alpha_G, \beta_G^G, \text{and} \beta_G^G\) have nice homotopical properties.

First we will deal with the maps \(\beta_G^G\) and \(\beta_G^G\). For example:

1. In (5.2.16) the map \(\beta_G^G : J^-[G] \rightarrow J[G]\) is the pushout product of two copies of the acyclic cofibration \(0 : \mathbb{I} \rightarrow J\), so it is also an acyclic cofibration by the pushout product axiom in \(M\).

2. In Example 5.2.17 with \(M = \text{Top}\) and \(J\) the unit interval \([0, 1]\), the map \(\beta_G^G : J^{n}[G] \rightarrow J[G]\) is the boundary inclusion of the square \([0, 1] \times 2\). So it is a cofibration, but not an acyclic cofibration.

We will formalize these properties in the next definition.

**Definition 7.2.5.** Suppose \(G\) is a \(C\)-colored unital pasting scheme, and \(J\) is a commutative interval in \(M\). We say that \(G\) is *compatible with \(J\)* if the following two conditions hold whenever \(G \in \mathcal{G}\) with \(|G| \geq 1\).

1. The map \(\beta_G : J^-[G] \rightarrow J[G]\) in (5.2.8) is an acyclic cofibration in \(M\).
(2) The map
\[ \beta_G^\circ : J^\circ[G] \longrightarrow J[G] \]
in \(5.2.9\) is a cofibration in \(M\).

We say that \(G\) is \textit{compatible with} \(J\) (resp., \(J^\circ\)) if condition (1) (resp., (2)) holds.

\textbf{Remark 7.2.6.} As we explained in the proof of Lemma \(5.2.11\) the map \(\beta_G^-(\text{resp., } \beta_G^\circ)\) involves only the map \(0 : 1 \longrightarrow J\) (resp., the maps \(0, 1 : 1 \longrightarrow J\)), but not the counit and the multiplication of \(J\). Therefore, compatibility of a pasting scheme \(G\) with \(J\) (resp., \(J^-\) and \(J^\circ\)) is independent of the multiplicative structure of the commutative segment.

The following observation shows that compatibility is automatically true in examples of interest.

\textbf{Proposition 7.2.7.} Suppose \(M\) is the model category of:
\begin{itemize}
  \item (pointed or unpointed) simplicial sets;
  \item compactly generated topological spaces;
  \item (bounded or unbounded) chain complexes or simplicial modules over a field of characteristic 0;
  \item small categories with the folk model structure.
\end{itemize}

Equipped \(M\) with the commutative interval \(J\) as in Example \(3.2.2\) Then every \(\mathcal{C}\)-colored unital pasting scheme \(G\) is compatible with \(J\).

\textbf{Proof.} Suppose \(M\) is the category of (pointed or unpointed) simplicial sets with the commutative interval
\[ \Delta^0 \sqcup \Delta^0 \xrightarrow{(0,1)} \Delta^1 \longrightarrow \Delta^0 \]
whose multiplication is given by the maximum operation on \(\{0, 1\}\). The other cases are proved similarly. In this case \(J[G]\) is a cube, and \(J^-[G]\) is a contractible subspace made up of some faces. Since \(\beta_G^-\) is an injection whose geometric realization is a weak equivalence, it is an acyclic cofibration. Likewise, \(\beta_G^\circ\) is a subspace inclusion into the cube \(J[G]\), so it is a cofibration. \(\Box\)

\textbf{Motivation 7.2.8.} To understand homotopical properties of the maps \(\text{colim} \delta_G^-\) and \(\text{colim} \delta_G^\circ\) in \(7.2.4\), we will need to understand homotopical properties of the maps \(\delta_G^-\) and \(\delta_G^\circ\) with respect to suitable automorphisms of a graph \(G\). These automorphisms are defined next.

\textbf{Definition 7.2.9.} Suppose \(G\) is a \(\mathcal{C}\)-colored unital pasting scheme, and \(G \in \mathcal{G}_{n+1}(\mathcal{C})\).

(1) Denote by \(\text{Aut}_G(K)\) the automorphism group of \(G\) in \(\mathcal{G}_{n+1}(\mathcal{C})\), i.e., the group of maps
\[ (C'_v) : G \longrightarrow G \]
in the substitution category \(\mathcal{G}_{n+1}(\mathcal{C})\) with each \(C'_v\) a permuted corolla.

(2) Denote by \(\mathcal{G}_{n+1}\) the groupoid of objects \(K \in \mathcal{G}\) with \(|K| = n + 1\). A map
\[ (C'_v : (C'_v')) : K' \longrightarrow K \]
in \(\mathcal{G}_{n+1}\) is a pair with:
\begin{itemize}
  \item \((C'_v) \in \text{Aut}_G(K)\);
  \item \(C'v\) a permuted corolla such that \(K' = C'(K(C'_v))\). 
\end{itemize}
Composition is given by graph substitution.

(3) Denote by \( \text{Aut}_w(G) \) the automorphism group of \( G \) in \( \mathcal{G}_{n+1} \).

It is important to note that maps in \( \text{Aut}_v(G) \) and \( \text{Aut}_w(G) \) do not permute the set of vertices in \( G \).

**Example 7.2.10.** Consider the following non-trivial automorphism in \( \text{Aut}_v(G) \) of a graph \( G \in \text{Gr}^1 \), the one-colored pasting scheme of connected wheel-free graphs:

\[
G = G(C_u \tau, \tau C_v) \xrightarrow{(C_u \tau, \tau C_v)} G
\]

At the vertices \( u \) and \( v \) in \( G \), the listings are as indicated, and \( C_u \) (resp., \( C_v \)) denotes the corolla with two inputs (resp., outputs). We write \( \tau \) for the non-identity permutation \((1 2)\) in \( \Sigma_2 \), so \( C_u \tau \) and \( \tau C_v \) are permuted corollas but not corollas. In the picture for \( C_u \tau \) (resp., \( \tau C_v \)), the numbers 1 and 2 indicate the graph listings, not vertex listings, of the respective input legs (resp., output legs). The graph substitution \( G(C_u \tau, \tau C_v) \) is the same as \( G \), so the map

\[
(C_u \tau, \tau C_v) : G \rightarrow G
\]

is a non-identity automorphism of \( G \) in \( \text{Aut}_v(G) \).

**Example 7.2.11.** Consider the following non-trivial automorphism in \( \text{Aut}_w(K) \) of a corolla \( K = C_{(0,2)} \in \text{Gr}^1 \), the one-colored pasting scheme of connected wheel-free graphs:

\[
K = (\tau C_v) K(\tau C_v)
\]

As in Example 7.2.10 \( \tau \) denotes the non-identity permutation \((1 2)\) in \( \Sigma_2 \), so \( \tau C_v \) is a permuted corolla but not a corolla. The number 1 or 2 at the end of an output leg indicates the graph listing, not the vertex listing. The graph substitution \( (\tau C_v) K(\tau C_v) \) is the same as the corolla \( K \), so the map

\[
(\tau C_v; \tau C_v) : K \rightarrow K
\]

is a non-identity automorphism of \( K \) in \( \text{Aut}_w(K) \).

**Definition 7.2.12.** Suppose \( \mathcal{G} \) is a \( \mathcal{C} \)-colored unital pasting scheme, and \( P \) is a \( \mathcal{G} \)-prop in \( \mathcal{M} \).

1. \( P \) is \( \Sigma_{\mathcal{C}} \)-cofibrant if its underlying \( \Sigma_{\mathcal{C}} \)-bimodule in \( \mathcal{M}^{S} \) is cofibrant.
(2) \( P \) is \( \Sigma^* \)-cofibrant if its underlying pointed \( \Sigma \)-bimodule in \( \mathcal{M}^{\Sigma^*} \) is cofibrant.

(3) \( P \) is \( G_0 \)-cofibrant if its underlying \( G_0 \)-prop (Def. 4.2.1) is cofibrant.

**Remark 7.2.13.** Note that \( G_0 \)-cofibrant implies \( \Sigma^* \)-cofibrant, which implies \( \Sigma^C \)-cofibrant. In fact, \( P \) is \( \Sigma^* \)-cofibrant if and only if:

1. It is \( \Sigma^C \)-cofibrant.
2. Each colored unit \( \gamma_P^{c} = 1_c : 1 \longrightarrow P(c) \) is a cofibration.

If \( \mathcal{G} \) is connected, then a \( \mathcal{G} \)-prop \( P \) is \( G_0 \)-cofibrant if and only if:

1. It is \( \Sigma^* \)-cofibrant.
2. The map \( \gamma_P^{c} = 0_c : 1 \longrightarrow P(c) \) is a cofibration for each \( c \in \mathcal{C} \) for which the exceptional loop \( Q_c \) belongs to \( \mathcal{G} \).

**Lemma 7.2.14.** Suppose \( P \) is a \( \Sigma^* \)-cofibrant \( \mathcal{G} \)-prop in \( \mathcal{M} \) with \( \mathcal{G} \) a \( \mathcal{C} \)-colored unital pasting scheme. Then for each \( G \in \mathcal{G} \) with \( |G| \geq 1 \), the map \( \alpha_G : P[G] \longrightarrow P[G] \) in (5.2.3) is an \( \text{Aut}_w(G) \)-cofibration (resp., \( \text{Aut}_v(G) \)-cofibration).

**Proof.** The iterated pushout product of the colored units of \( P \) corresponding to the tunnels in \( G \) is the map

\[
\colim_{v \in \text{Tun}(G)} \left( \bigotimes_{v \in \text{Tun}(G)} P(v) \right) \xrightarrow{\alpha_G'} \bigotimes_{v \in \text{Tun}(G)} P(v)
\]

in \( \mathcal{M}^{\text{Aut}_w(G)} \). Since each colored unit of \( P \) is a cofibration in \( \mathcal{M} \), the map \( \alpha_G' \in \mathcal{M}^{\text{Aut}_w(G)} \) is an underlying cofibration in \( \mathcal{M} \) by repeated applications of the pushout product axiom. The map \( \alpha_G \) decomposes as

\[
\alpha_G = \left( \bigotimes_{v \in G \setminus \text{Tun}(G)} P(v) \right) \otimes \alpha_G'
\]

in which the object \( \bigotimes_{v \in G \setminus \text{Tun}(G)} P(v) \) is \( \text{Aut}_w(G) \)-cofibrant because \( P \) is \( \Sigma^e \)-cofibrant. So \( \alpha_G \) is an \( \text{Aut}_w(G) \)-cofibration by \( \text{BM06} \) Lemma 2.5.2. The proof for the \( \text{Aut}_v(G) \) case is the same. \( \square \)

**Motivation 7.2.16.** For a nice enough \( \mathcal{G} \)-prop \( P \) in \( \mathcal{M} \), we want to show that the augmentation \( \eta : W(\mathcal{G}, J, P) \longrightarrow P \) is a cofibrant resolution of \( P \). In particular, we need to show:

1. The \( W \)-construction \( W(\mathcal{G}, J, P) \) is a cofibrant \( \mathcal{G} \)-prop.
2. The augmentation \( \eta \) is a weak equivalence of \( \mathcal{G} \)-props.

The next observation deals with the second statement.

**Theorem 7.2.17.** Suppose \( J \) is a commutative interval in \( \mathcal{M} \) with \( 1 \) cofibrant, and \( P \) is a \( \Sigma^* \)-cofibrant \( \mathcal{G} \)-prop in \( \mathcal{M} \) with \( \mathcal{G} \) a \( \mathcal{C} \)-colored connected unital pasting scheme compatible with \( J^- \). Then the augmentation

\[
\eta : W(\mathcal{G}, J, P) \longrightarrow P
\]

in Prop. 4.1.2 is a weak equivalence of \( \mathcal{G} \)-props.
induced map $C$ that the map $M$ in intervals is an acyclic cofibration for $n$ and is a weak equivalence for each pair of $\mathcal{C}$-profiles. By the filtration in Prop. 5.1.15 and (10.3.4), it suffices to show that each map

$$W(\mathcal{G}, J, P)_{n-1}(\xi) \longrightarrow W(\mathcal{G}, J, P)_{n}(\xi)$$

is an acyclic cofibration for $n \geq 0$. By the pushout diagram in Theorem 5.4.7, it is enough to show that the map

$$\colim_{G \in \mathcal{G}_{n+1}(\xi)} (J \otimes P)^{-}[G] = \coprod_{\{G\}_{w} \in \mathcal{G}_{n+1}(\xi)} \left[ \colim_{G \in \mathcal{G}[G]} (J \otimes P)^{-}[G] \right]$$

is an acyclic cofibration. So it suffices to show that, for each maximal connected sub-groupoid $[G]_w$ in $\mathcal{G}_{n+1}(\xi)$, the map

$$\colim_{G \in \mathcal{G}[G]} (J \otimes P)^{-}[G] \colim_{G \in \mathcal{G}[G]} \delta_G \colim_{G \in \mathcal{G}[G]} (J \otimes P)[G]$$

is an acyclic cofibration, where $\delta_G$ is the pushout corner map $\alpha_G \sqcup \beta_G$. We have that

$$\colim \delta_G \cong (\delta_G)_{\Aut(G)} ,$$

where $(?)_{\Aut(G)}$ means taking $\Aut(G)$-coinvariants. The map

$$\alpha_G : \mathcal{P}^{-}[G] \longrightarrow \mathcal{P}[G]$$

is an $\Aut(G)$-cofibration by Lemma 7.2.14. The map

$$\beta_G : J^{-}[G] \longrightarrow J[G] \in \mathcal{M}^{\Aut(G)}$$

is an underlying acyclic cofibration in $\mathcal{M}$ by the compatibility assumption. Therefore, by Lemma 2.5.2, their pushout corner map $\delta_G = \alpha_G \sqcup \beta_G$ is an $\Aut(G)$-acyclic cofibration. Since taking $\Aut(G)$-coinvariants is a left Quillen functor, $(\delta_G)_{\Aut(G)}$ is an acyclic cofibration in $\mathcal{M}$. 

**Motivation 7.2.18.** In Corollary 4.3.2 we observed that the construction $W(\mathcal{G}, J, P)$ is natural with respect to changing the commutative segment $J$ and the $\mathcal{G}$-prop $P$. These are purely categorical properties. In a homotopical setting, the next two observations say that the $W$-construction is a homotopy invariant in both the $J$ and the $P$ variables.

**Corollary 7.2.19.** Suppose $J \longrightarrow J'$ is a weak equivalence of commutative intervals in $\mathcal{M}$ with $\mathcal{I}$ cofibrant, and $P$ is a $\Sigma_{w}$-cofibrant $\mathcal{G}$-prop in $\mathcal{M}$ with $\mathcal{G}$ a $\mathcal{E}$-colored connected unital pasting scheme compatible with $J^{-}$ and $J'^{-}$. Then the induced map

$$W(\mathcal{G}, J, P) \longrightarrow W(\mathcal{G}, J', P)$$
7.3. Cofibrant Resolution of Generalized Props

**Lemma 7.3.1.** Suppose \( J \) is a commutative interval in \( \mathcal{M} \) with \( 1 \) cofibrant, and \( P \xrightarrow{} Q \) is a weak equivalence of \( \Sigma^- \)-cofibrant \( G \)-props in \( \mathcal{M} \) with \( G \) a \( \mathcal{C} \)-colored connected unital pasting scheme compatible with \( J \). Then for each \( n \geq 0 \) the map
\[
\mathcal{W}(G, J, P) \xrightarrow{\delta_G} \mathcal{W}(G, J, P)_{n+1}
\]
in Prop. 5.5.3 is a cofibration of \( \Sigma\mathcal{C} \)-bimodules in \( \mathcal{M} \).

**Proof.** The right pushout diagram in (5.5.3) induces a pushout diagram in \( \Sigma\mathcal{C} \)-bimodules. Similar to the proof of Theorem 7.2.17, it suffices to show that the map
\[
(J \otimes P)^n[G] \xrightarrow{\delta_G} (J \otimes P)[G]
\]
is an \( \text{Aut}_w(G) \)-cofibration for each \( G \in \mathcal{G}_{-n+1}(\mathcal{C}) \), where \( \delta_G \) is the pushout product map \( \alpha_G \sqcup \beta_G^\circ \). The map
\[
\alpha_G : P^{-}[G] \longrightarrow P[G]
\]
is an \( \text{Aut}_w(G) \)-cofibration by Lemma 7.2.14. The map
\[
\beta_G^\circ : J^o[G] \longrightarrow J[G] \in \mathcal{M}_{\text{Aut}_w(G)}
\]
is an underlying cofibration in \( \mathcal{M} \) by the compatibility assumption. Therefore, by [BM06] Lemma 2.5.2, the pushout product map \( \delta_G^\circ \) is an \( \text{Aut}_w(G) \)-cofibration.

We now have all the ingredients to prove our main homotopical result. Namely, for a nice enough \( G \)-prop \( P \), the augmentation \( \eta : \mathcal{W}(G, J, P) \longrightarrow P \) is a cofibrant resolution of \( P \).

**Theorem 7.3.2.** Suppose \( J \) is a commutative interval in \( \mathcal{M} \) with \( 1 \) cofibrant, and \( P \) is a \( \mathcal{G}_0 \)-cofibrant \( G \)-prop in \( \mathcal{M} \) with \( G \) a \( \mathcal{C} \)-colored connected unital pasting scheme compatible with \( J \). Then the augmentation
\[
\eta : \mathcal{W}(G, J, P) \longrightarrow P
\]
is a cofibrant resolution of the \( G \)-prop \( P \).
Proof. The map $\eta$ is a weak equivalence of $G$-props by Theorem 7.2.17. It remains to show that $W(G, J, P)$ is a cofibrant $G$-prop. Since $P$ is $G_0$-cofibrant and since the left adjoint $F^G$ (Lemma 4.2.9) is a left Quillen functor from $G_0$-props to $G$-props in $M$, it follows that $F^G P$ is a cofibrant $G$-prop, in which the forgetful functor $U$ has been omitted from the notation. So it suffices to show that the map 

$$\delta : F^G P \longrightarrow W(G, J, P)$$

in Theorem 4.2.14 is a cofibration of $G$-props.

Suppose given a solid-arrow commutative diagram

$$\begin{array}{ccc}
F^G P & \xrightarrow{\phi_0} & Q \\
\downarrow \delta & & \downarrow x \\
W(G, J, P) & \xrightarrow{\psi} & T
\end{array}$$

in $G$-props with $\chi$ an acyclic fibration. We must show that there exists a dotted filler $\phi$ that makes the whole diagram commutative. By adjunction it is enough to show that the adjoint diagram

$$(7.3.3) 
\begin{array}{ccc}
W(G, J, P)_0 \cong P & \xrightarrow{\phi_0} & Q \\
\downarrow \phi & & \downarrow x \\
W(G, J, P) & \xrightarrow{\psi} & T
\end{array}$$

in $G_0$-props has a dotted filler $\psi$ that is a map of $G$-props.

Since $\phi_0$ is a map of $G_0$-props, it is also a 0-map (Def. 6.1.2). Inductively, suppose we have constructed an $n$-map

$$\{ \phi_k : W(G, J, P)_k \longrightarrow Q \}_{0 \leq k \leq n}$$

such that the solid-arrow diagram

$$\begin{array}{ccc}
W(G, J, P)_0 \cong P & \xrightarrow{\phi_0} & Q \\
\downarrow \phi_n & & \downarrow x \\
W(G, J, P) & \xrightarrow{\psi} & T
\end{array}$$

of $\Sigma_e$-bimodules is commutative. By Theorem 6.2.2 there is a canonical extension of $\phi_n$ to $\phi_n^\ast$. Since $\chi$ is an acyclic fibration of $\Sigma_e$-bimodules, we may extend $\phi_n^\ast$ to $\phi_{n+1}$ by Lemma 7.3.1. Then $\{ \phi_k \}_{0 \leq k \leq n+1}$ is an $(n+1)$-map by Theorem 6.3.2. So by induction on $n$ and Prop. 6.1.4, the desired filler $\phi$ exists in (7.3.3). □

Motivation 7.3.4. In Theorem 4.4.3 we observed that the $W$-construction is well-behaved with respect to changing the base categories. The next result says that the $W$-construction is well-behaved with respect to left Quillen functors from
the base category. For a $G$-prop $P$ in $M$ and a nice enough left Quillen functor $f : M \to N$, we can resolve the $G$-prop $fP$ in $N$ in two ways. First, we can use the $W$-construction of $fP$ in $N$. Second, we can apply $f$ to the $W$-construction of $P$ in $M$. The next result says that these resolutions are connected by a weak equivalence.

**Theorem 7.3.5.** Suppose $f : M \to N$ is a unit-preserving lax symmetric monoidal, left Quillen functor between cofibrantly generated monoidal model categories with cofibrant monoidal units. Suppose the colored operad for $G$-props is admissible in both $M$ and $N$ for some $\mathcal{C}$-colored connected unital pasting scheme $G$ that is compatible with a commutative interval $J$ in $M$ and with $(fJ)^{-}$ in $N$. Suppose that cofibrant $G$-props in $M$ are also entrywise cofibrant in $M$. Then for each $G_0$-cofibrant $G$-prop $P$ in $M$, the map

$$f_* : W(G, fJ, fP) \to fW(G, J, P)$$

in Theorem 4.4.3 is a weak equivalence of $G$-props in $N$.

**Proof.** Using the commutative diagram (4.4.4) and the 2-out-of-3 property, it suffices to show that $\eta f^P$ and $f \eta^P$ are entrywise weak equivalences in $N$.

Since $P$ is in particular $\Sigma_\ast$-cofibrant (Remark 7.2.13), the $G$-prop $fP$ in $N$ is also $\Sigma_\ast$-cofibrant by [Hir03] (Theorem 11.6.5(1)). Since $fJ$ is a commutative interval in $N$, Theorem 7.2.17 implies that the augmentation

$$\eta f^P : W(G, fJ, fP) \to fP$$

is a weak equivalence of $G$-props in $N$.

By Theorem 7.3.2 the augmentation

$$\eta^P : W(G, J, P) \to P$$

is a weak equivalence of $G$-props in $M$ with a cofibrant domain, hence also entrywise cofibrant in $M$ by assumption. Since $P$ is $\Sigma$-cofibrant, it is also entrywise cofibrant in $M$ by [Hir03] (Prop. 11.6.3). By Ken Brown’s Lemma [Hir03] (Cor. 7.7.2(1)) $f\eta^P$ is entrywise a weak equivalence in $N$. $\square$

**Example 7.3.6.** The model categories of (pointed or unpointed) simplicial sets, of (bounded or unbounded) chain complexes or simplicial modules over a field of characteristic 0, and of small categories with the folk model structure satisfy all the conditions for $M$ and $N$ in Theorem 7.3.5. As mentioned before, the admissibility of the colored operad for $G$-props is proved in [WY17] (Section 8). Compatibility with commutative intervals is Prop. 7.2.7. Every object in each of these model categories is cofibrant. For example, Theorem 7.3.5 applies to the left Quillen functor from simplicial sets to small categories that takes a simplicial set to its fundamental groupoid [Rez]. It also applies to the normalized chain functor from simplicial modules to non-negatively graded chain complexes over a field of characteristic 0, which is part of the Dold-Kan correspondence [Dol58, Kan58].

### 7.4. Cofibrancy With Respect To Changing Intervals

**Definition 7.4.1.** Suppose $G$ is a $\mathcal{C}$-colored unital pasting scheme, and $f : J \to J'$ is a map of commutative intervals that is a cofibration in $M$. We say that
\( G \) is \emph{compatible with} \( f \) if the pushout corner map \( f_G \) in the diagram

\[
\begin{align*}
\xymatrix{ J^0[G] & J[G] \\
\downarrow f^s & \downarrow f_G \\
\downarrow \beta^G & \downarrow \beta^G \\
J^0[G] & \coprod_{J^0[G]} J[G] & J'[G] \\
\downarrow f^s & \downarrow f_G & \downarrow f_G \\
\downarrow \beta^G & \downarrow \beta^G & \downarrow \beta^G \\
J^0[G] & \coprod_{J^0[G]} J[G] & J'[G]
}\end{align*}
\]

is a cofibration in \( \mathcal{M} \) whenever \( G \in \mathcal{G} \) with \( |G| \geq 1 \), where \( \beta^G \) is defined in (5.2.9).

**Example 7.4.3.** Suppose \( \mathcal{M} \) is the model category of (pointed or unpointed) simplicial sets. Then

\[
f = (d^0, d^1) : \Delta^0 \cup \Delta^0 \longrightarrow \Delta^1
\]

is a map of commutative intervals that is a cofibration in \( \mathcal{M} \). The pushout corner map \( f_G \) is an injection, hence a cofibration. So every \( \mathcal{E} \)-colored unital pasting scheme is compatible with \( f \). The obvious analogues for the model categories of (bounded or unbounded) chain complexes over a field of characteristic 0 and of small categories with the folk model structure also hold.

**Theorem 7.4.4.** Suppose \( f : J \longrightarrow J' \) is a map of commutative intervals in \( \mathcal{M} \) with \( 1 \) cofibrant and with \( f \) a cofibration (resp., an acyclic cofibration) in \( \mathcal{M} \). Suppose \( P \) is a \( \Sigma_* \)-cofibrant \( G \)-prop in \( \mathcal{M} \) with \( G \) a \( \mathcal{E} \)-colored connected unital pasting scheme compatible with \( f \) (resp., \( f^s \), \( J^s \), and \( J'^s \)). Then the induced map

\[
W(G, J, P) \longrightarrow W(G, J', P)
\]

is a cofibration (resp., an acyclic cofibration) of \( G \)-props.

**Proof.** Using Cor. [7.2.19] it suffices to prove the case when \( f \) is a cofibration in \( \mathcal{M} \) and when \( G \) is compatible with \( f \). Suppose given a commutative solid-arrow diagram

\[
\xymatrix{ W(G, J, P) & Q \\
W(G, J', P) & T \\
\downarrow f^s & \downarrow \chi \\
\downarrow \phi & \downarrow \psi \\
Q & T
}\]

of \( G \)-props with \( \chi \) an acyclic fibration. We must show that there exists a dotted filler \( \phi \) that makes the entire diagram commutative. By Prop. [6.1.4] it is enough to construct maps

\[
\phi_k : W(G, J', P)_k \longrightarrow Q
\]
of $\Sigma_\mathcal{C}$-bimodules for $k \geq 0$ such that $\{\phi_k\}_{0 \leq k \leq n}$ is an $n$-map and that the diagram

$$\begin{array}{c}
\begin{array}{c}
\xymatrix{
W(\mathcal{G}, J, P)_n \ar[r]^-{\varphi_n} \ar[d]_-{f_*} & W(\mathcal{G}, J, P)_n^\circ \ar[r]^-{\varphi_{n+1}} \ar[d]_-{\psi_n} & Q \ar[d]^-{\chi} \\
W(\mathcal{G}, J', P)_n \ar[r]^-{\varphi_n} & W(\mathcal{G}, J', P)_n^\circ \ar[r]^-{\psi_{n+1}} & Q
}
\end{array}
\end{array}$$

of $\Sigma_\mathcal{C}$-bimodules is commutative for each $n \geq 0$. Here $\varphi_n$ (resp., $\psi_n$) is the restriction of $\varphi$ (resp., $\psi$) to $W(\mathcal{G}, J, P)_n$ (resp., $W(\mathcal{G}, J', P)_n$). By Lemma 5.1.13 we define $\phi_0$ to be the restriction of $\varphi$ to

$$W(\mathcal{G}, J, P)_0 \cong P \cong W(\mathcal{G}, J', P)_0.$$ 

Inductively, suppose we have constructed an $n$-map $\{\phi_k\}_{0 \leq k \leq n}$ such that (7.4.5) is commutative. To extend this to the desired map $\phi_{n+1}$, first note that by Theorem 6.2.2 there is a canonical extension of $\phi_n$ to a map

$$\phi_n^0 : W(\mathcal{G}, J', P)_n^\circ \to Q$$

of $\Sigma_\mathcal{C}$-bimodules such that $\chi \phi_n^0$ is the restriction of $\psi$ to $W(\mathcal{G}, J', P)_n^\circ$. Furthermore, by the definition of $\phi_n^0$ (6.2.4), for each object

$$(H_v) : G \to K \in D^\circ(G)$$

with $|G| = n + 1$ (so $|K| > 1$ and $\bigsqcup_{v \in K} |H_v| < |G|$), the diagram

$$\begin{array}{c}
\xymatrix{
\bigotimes_{v \in K} (J \otimes P)[H_v] \ar[r]^-{\otimes\frac{1}{|K|}} \ar[d]_-{\otimes f} & (J \otimes P)[G] \ar[r]^-{\omega_G} \ar[d]_-{\otimes f_*} & W(\mathcal{G}, J, P)_n^\circ \ar[r]^-{\varphi} \ar[d]_-{\otimes f_*} & \bigotimes_{v \in K} W(H_v)(v) \ar[r]^-{\otimes f_*} \ar[d]_-{\otimes f_*} & Q(v) = Q[K] \\
\bigotimes_{v \in K} \omega_{H_v} \ar[r]^-{\omega_{H_v}} \ar[d]_-{\otimes f_*} & \bigotimes_{v \in K} W[\mathcal{G}, J, P]_n \ar[r]^-{\varphi} \ar[d]_-{\otimes f_*} & W[\mathcal{G}, J, P]_n \ar[r]^-{\varphi} \ar[d]_-{\otimes f_*} & \bigotimes_{v \in K} W[\mathcal{G}, J, P]_n \ar[r]^-{\varphi} \ar[d]_-{\otimes f_*} & Q(v) = Q[K]
}
\end{array}$$

is commutative, in which

$$W = W(\mathcal{G}, J, P) \quad \text{and} \quad W' = W(\mathcal{G}, J', P).$$

The top trapezoid is the definition of $\gamma^W_\mathcal{G}$ (3.5.6). The right triangle is commutative because $\varphi$ is a map of $\mathcal{G}$-props. The commutativity of the diagram (7.4.6) implies that $\phi_n^0 f_*$ is the restriction of $\varphi$ to $W(\mathcal{G}, J, P)_n^\circ$.

By Theorem 6.3.2, if there is an extension of $\phi_n^0$ to $\phi_{n+1}$, then $\{\phi_k\}_{0 \leq k \leq n+1}$ is an $(n + 1)$-map. We need such an extension $\phi_{n+1}$ with the property that

$$\phi_{n+1} f_* = \varphi_{n+1}.$$
Since $\chi$ is an acyclic fibration, it suffices to show that the pushout corner map $\overline{f}$ in the diagram

\[
\begin{array}{ccc}
W(G,J,P)_n & \rightarrow & W(G,J,P)_{n+1} \\
\downarrow & \Downarrow & \downarrow \\
W(G',J',P)_n & \rightarrow & W(G',J',P)'_n \\
\end{array}
\]

is a cofibration of $\Sigma e$-bimodules.

There is a commutative cube

\[
\begin{array}{ccc}
\text{colim} \ (J \otimes P)[G] & \rightarrow & \text{colim} \ (J \otimes P)[G] \\
\downarrow & \Downarrow & \downarrow \\
W(G,J,P)_n & \rightarrow & W(G,J,P)_{n+1} \\
\text{colim} \ (J' \otimes P)[G] & \rightarrow & \text{colim} \ (J' \otimes P)[G] \\
\downarrow & \Downarrow & \downarrow \\
W(G',J',P)_n & \rightarrow & W(G',J',P)'_n \\
\end{array}
\]

of $\Sigma e$-bimodules such that the top and the bottom faces are pushouts (5.5.3), that all vertical maps are induced by $\iota$, and that $G_{n+1}$ is the groupoid in Def. 7.2.9. By [BM06] Lemma 6.9, there is a pushout diagram

\[
\begin{array}{ccc}
\text{colim} \ (J' \otimes P)[G] & \rightarrow & \text{colim} \ (J \otimes P)[G] \\
\downarrow & \Downarrow & \downarrow \\
W(G,J,P)_n & \rightarrow & W(G,J,P)_{n+1} \\
\end{array}
\]

of $\Sigma e$-bimodules in which each colimit in the top row is indexed by $G \in G_{n+1}$, and the top (resp., bottom) horizontal map $f_*$ (resp., $\overline{f}$) is the pushout corner map of the back (resp., front) face of the above cube. So to show that $\overline{f}$ is a cofibration, it suffices to show that $f_*$ in (7.4.8) is a cofibration of $\Sigma e$-bimodules.
For each $G \in \mathcal{G}$, consider the pushout corner map $f'_G$ below.

\[
\begin{array}{c}
(J \otimes P)^* [G] \quad \rightarrow \quad (J \otimes P)[G] \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
(J' \otimes P)^* [G] \quad \rightarrow \quad (J' \otimes P)[G] \\
\bigcup_{(J \otimes P)^* [G]} \quad \rightarrow \quad (J \otimes P)[G] \\
\end{array}
\]

Then the map $f_*$ in (7.4.8) is

\[
f_* = \bigcup_{[G] \in \mathcal{G}_{n+1}} \text{colim} f'_G
\]

with $[G]$ the maximal connected sub-groupoid of $\mathcal{G}_{n+1}$ containing $G$. To show that $f_*$ in (7.4.8) is a cofibration of $\Sigma^*$-bimodules, it suffices to show that each $f'_G$ is an $\text{Aut}_w(G)$-cofibration, where $\text{Aut}_w(G)$ is the automorphism group of $G \in \mathcal{G}_{n+1}$ (Def. 7.2.9).

For each $G \in \mathcal{G}$ with $|G| = n + 1$, the map $f'_G$ is also the pushout product map in the diagram

\[
\begin{array}{c}
\left( J'^*[G] \bigcup_{J^*[G]} J[G] \right) \otimes P^-[G] \quad \rightarrow \quad J'[G] \otimes P^-[G] \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\left( J'^*[G] \bigcup_{J^*[G]} J[G] \right) \otimes P[G] \quad \rightarrow \quad (J' \otimes P)[G] \\
\quad \quad \uparrow \quad \quad \uparrow \quad \quad \uparrow \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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\quad \quad \quad \quad \quad \quad \quad \quad \quad \qa
PROOF. By Cor. 7.2.20, it suffices to consider the case when $G$ is compatible with $J^\circ$ and $g$ is a $G_0$-cofibration. Suppose given a commutative solid-arrow diagram

$$
\begin{array}{ccc}
W(G, J, P) & \overset{\varphi}{\longrightarrow} & R \\
g_*\downarrow & & \downarrow \chi \\
W(G, J, Q) & \overset{\psi}{\longrightarrow} & T
\end{array}
$$

of $G$-props with $\chi$ an acyclic fibration. We must show that there exists a dotted filler $\phi$ that makes the entire diagram commutative. As in the proof of Theorem 7.4.4, by Prop. 6.1.4, it is enough to construct maps

$$
\phi_k : W(G, J, Q)_k \longrightarrow R
$$

of $\Sigma_{\mathcal{C}}$-bimodules for $k \geq 0$ such that $\{\phi_k\}_{0 \leq k \leq n}$ is an $n$-map and that the diagram

$$
\begin{array}{ccc}
W(G, J, P)_n & \longrightarrow & W(G, J, P)^\circ_n \\
\varphi_n & \downarrow \phi_n & \downarrow \chi \\
W(G, J, Q)_n & \longrightarrow & W(G, J, Q)^\circ_n
\end{array}
$$

is commutative for each $n \geq 0$. The required 0-map $\phi_0$ exists by Lemma 5.1.13, $\chi$ an acyclic fibration, and the $G_0$-cofibration assumption of $g$.

Inductively, suppose we have constructed an $n$-map $\{\phi_k\}_{0 \leq k \leq n}$ such that (7.5.2) is commutative. Following the proof of Theorem 7.4.4, to show that the required map $\phi_{n+1}$ exists, it is enough to show that the pushout corner map $\varpi$ in the diagram

$$
\begin{array}{ccc}
W(G, J, P)^\circ_n & \longrightarrow & W(G, J, P)_{n+1} \\
g_*\downarrow & \text{pushout} & \downarrow g_* \\
W(G, J, Q)^\circ_n & \longrightarrow & W(G, J, Q)_{n+1}
\end{array}
$$

is a cofibration of $\Sigma_{\mathcal{C}}$-bimodules.
7.5. COFIBRANCY WITH RESPECT TO CHANGING GENERALIZED PROPS

There is a commutative cube

$$\begin{array}{ccc}
\colim_{G \in \mathcal{G}_{n+1}} (J \otimes P)^{\circ}[G] & \rightarrow & \colim_{G \in \mathcal{G}_{n+1}} (J \otimes P)[G] \\
\downarrow & & \downarrow \\
W(G, J, P)^{\circ} & \rightarrow & W(G, J, P)_{n+1} \\
\colim_{G \in \mathcal{G}_{n+1}} (J \otimes Q)^{\circ}[G] & \rightarrow & \colim_{G \in \mathcal{G}_{n+1}} (J \otimes Q)[G] \\
\downarrow & & \downarrow \\
W(G, J, Q)^{\circ} & \rightarrow & W(G, J, Q)_{n+1}
\end{array}$$

of $\Sigma \mathcal{E}$-bimodules such that the top and the bottom faces are pushouts (5.5.3), that all vertical maps are induced by $g$, and that $\mathcal{G}_{n+1}$ is the groupoid in Def. 7.2.9.

By [BM06] Lemma 6.9, there is a pushout diagram

$$\begin{array}{ccc}
\colim(J \otimes Q)^{\circ}[G] \coprod_{\colim(J \otimes P)^{\circ}[G]} \colim(J \otimes Q)[G] & \rightarrow & \colim(J \otimes Q)[G] \\
\downarrow & & \downarrow \\
W(G, J, Q)^{\circ} \coprod_{W(G, J, P)^{\circ}} W(G, J, P)_{n+1} & \rightarrow & W(G, J, Q)_{n+1}
\end{array}$$

of $\Sigma \mathcal{E}$-bimodules in which each colimit in the top row is indexed by $G \in \mathcal{G}_{n+1}$, and the top (resp., bottom) horizontal map $g_*$ (resp., $\overline{g}$) is the pushout corner map of the back (resp., front) face of the above cube. So to show that $g_*$ is a cofibration, it suffices to show that $g_*$ in (7.5.4) is a cofibration of $\Sigma \mathcal{E}$-bimodules.

For each $G \in \mathcal{G}$, consider the pushout corner map $g'_G$ below.

$$\begin{array}{ccc}
(J \otimes P)^{\circ}[G] & \rightarrow & (J \otimes P)[G] \\
\downarrow & & \downarrow \\
(J \otimes Q)^{\circ}[G] & \rightarrow & (J \otimes Q)^{\circ}[G] \coprod_{(J \otimes P)^{\circ}[G]} (J \otimes P)[G] \\
\downarrow & & \downarrow \\
& \rightarrow & (J \otimes Q)[G]
\end{array}$$

Then the map $g_*$ in (7.5.4) is

$$g_* = \coprod_{[G] \in \mathcal{G}[G]} \colim_{G \in \mathcal{G}} g'_G$$

with $[G]$ the maximal connected sub-groupoid of $\mathcal{G}_{n+1}$ containing $G$. To show that $g_*$ in (7.5.4) is a cofibration of $\Sigma \mathcal{E}$-bimodules, it suffices to show that each $g'_G$ is an $\text{Aut}_w(G)$-cofibration, where $\text{Aut}_w(G)$ is the automorphism group of $G \in \mathcal{G}_{n+1}$ (Def. 7.2.9).
For each $G \in \mathcal{G}$ with $|G| = n + 1$, the map $g'_G$ is also the pushout product map $\beta^\circ_G \square \theta_G$ in the diagram
\[
\begin{array}{c}
J^\circ[G] \otimes \left( P[G] \coprod_{P^-[G]} Q^-[G] \right) \\
\downarrow \downarrow \downarrow \downarrow \downarrow
\end{array}
\begin{array}{c}
J^\circ[G] \otimes Q[G] \\
\downarrow \downarrow \downarrow \downarrow
\end{array}
\begin{array}{c}
Z \\
\downarrow \downarrow \downarrow \downarrow
\end{array}
\begin{array}{c}
(J \otimes Q)[G] \\
\downarrow \downarrow \downarrow \downarrow
\end{array}
\]
with $\beta^\circ_G$ the map in (5.2.9) and $\theta_G$ the pushout corner map in the following diagram.

\[
\begin{array}{c}
P^-[G] \xrightarrow{g_*} Q^-[G] \\
\downarrow \downarrow \downarrow \downarrow
\end{array}
\begin{array}{c}
P[G] \coprod_{P^-[G]} Q^-[G] \\
\downarrow \downarrow \downarrow \downarrow
\end{array}
\begin{array}{c}
\theta_G \\
\downarrow \downarrow \downarrow \downarrow
\end{array}
\begin{array}{c}
Q[G] \\
\downarrow \downarrow \downarrow \downarrow
\end{array}
\]

Since $\beta^\circ_G \in \mathcal{M}^{\text{Aut}_w(G)}$ is an underlying cofibration in $\mathcal{M}$ by the compatibility assumption, to show that $g'_G = \beta^\circ_G \square \theta_G$ is an $\text{Aut}_w(G)$-cofibration, it suffices to show that $\theta_G$ is an $\text{Aut}_w(G)$-cofibration by [BM06] Lemma 2.5.2. This is true by Lemma 7.5.6 below.

**Lemma 7.5.6.** The map $\theta_G$ in (7.5.5) is an $\text{Aut}_w(G)$-cofibration.

**Proof.** Recall that the map $\alpha_G$ (5.2.3) is the identity on non-tunnel vertices. The map $\theta_G$ can be rewritten as
\[
\theta_G = \left( \bigotimes_{v \in G \setminus \text{Tun}(G)} Q(v) \right) \otimes \theta'_G
\]
in which $\text{Tun}(G)$ is the set of all the tunnels in $G$, and $v$ runs through all the vertices in $G$ that are not tunnels. The map $\theta'_G$ is defined as the pushout corner map in the diagram
in which the maps $\alpha'_G$ are defined in (7.2.15).

Since $Q$ is assumed $\Sigma_*$-cofibrant, the object $\mathcal{O}_v Q(v)$ in (7.5.7) is $\text{Aut}_w(G)$-cofibrant. By [BM06] Lemma 2.5.2 it remains to show that $\theta'_G \in M^{\text{Aut}_w(G)}$ is an underlying cofibration in $M$. The map $\theta'_G$ is a cofibration in $M$ by [BM06] Lemma 2.5.4, using the cofibrations

\[
1 \longrightarrow P(v) \longrightarrow^g Q(v)
\]

for all $v \in \text{Tun}(G)$.

$\square$
Part 2

Applications and Related Constructions
Applications

In this chapter we discuss applications of some of the results in the previous chapters. In Section 8.1 we discuss cofibrant resolutions of properads and wheeled operads in chain complexes. In particular, this leads to a simple definition of an \(A_\infty\)-bialgebra as an algebra over the \(W\)-construction of the bialgebra properad. In Section 8.2 we discuss rectification and homotopy invariance of operad algebras. In particular, when applied to the commutative operad in chain complexes, the \(W\)-construction is a cofibrant \(E_\infty\)-operad. In Section 8.3 we discuss homotopy coherent diagrams. In Section 8.4 we discuss cofibrant resolutions of graphically generated generalized props.

8.1. Resolutions of Chain Generalized Props

Throughout this section, suppose \(M\) is the model category of (bounded or unbounded) chain complexes over a field of characteristic 0, equipped with the commutative interval \(J = N\Delta^1\), i.e., the normalized chain of the standard simplicial interval. By Prop. 7.2.7 every unital pasting scheme is compatible with \(J\). For each \(C\)-colored unital pasting scheme \(G\), by Maschke’s Theorem all \(G\)-props are \(\Sigma_C\)-cofibrant.

8.1.1. Resolutions of Chain Wheeled Operads. Suppose \(G = \text{Tree}^\circ\) is the one-colored pasting scheme of wheeled trees, so \(G\)-props in \(M\) are one-colored chain wheeled operads. Being \(G_0\)-cofibrant means that the structure maps

\[
\gamma_1 : 1 \to P(1) \quad \text{and} \quad \gamma_0 : 1 \to P(0)
\]

are cofibrations in \(M\), i.e., injections.

The forgetful functor from wheeled operads to operads admits a left adjoint

\[
L : \text{Operad} \to \text{Operad}^\circ,
\]

which is the special case of Theorem 1.7.1 for the inclusion

\[
\text{UTree} \leq \text{Tree}^\circ.
\]

For an operad \(P\), we will write \(LP\) as \(P^\circ\), called the \textit{wheeled completion} of \(P\) in [MMS09]. An operad \(P\) is said to be \textit{well-pointed} if the unit map

\[
\gamma_1 : 1 \to P(1)
\]

is a cofibration.

Corollary 8.1.1. Suppose \(P\) is a well-pointed operad in \(M\). Then the augmentation

\[
\eta : W(\text{Tree}^\circ, N\Delta^1, P^\circ) \to P^\circ
\]

in Prop. 4.1.1 is a cofibrant resolution of the wheeled operad \(P^\circ\).
PROOF. In the wheeled completion $P^\circ$, the structure map $\gamma^\circ$ is freely added to $P$, hence is a cofibration. By the remarks above $P^\circ$ is a $G_0$-cofibrant wheeled operad. So the assertion follows from Theorem 7.3.2. □

**Example 8.1.2.** Corollary 8.1.1 applies when $P$ is:

1. the associative operad $\text{As}$, whose algebras are unital differential graded (dg) algebras;
2. the commutative operad $\text{Com}$, whose algebras are unital commutative dg algebras;
3. the standard dg $A_\infty$-operad [Mar96] (Section 3), whose algebras are dg $A_\infty$-algebras;
4. the dg Barratt-Eccles $E_\infty$-operad [BE74, BF04], whose algebras are dg $E_\infty$-algebras.

So $W\left(\text{Tree}^\circ, N\Delta^1, \text{As}^\circ\right) \xrightarrow{\eta} \text{As}^\circ$, $W\left(\text{Tree}^\circ, N\Delta^1, A_\infty^\circ\right) \xrightarrow{\eta} A_\infty^\circ := (A_\infty)^\circ$, $W\left(\text{Tree}^\circ, N\Delta^1, \text{Com}^\circ\right) \xrightarrow{\eta} \text{Com}^\circ$, $W\left(\text{Tree}^\circ, N\Delta^1, E_\infty^\circ\right) \xrightarrow{\eta} E_\infty^\circ := (E_\infty)^\circ$ are cofibrant resolutions of wheeled operads. In [MMS09] different resolutions, called the wheeled minimal resolutions, of $\text{As}^\circ$ and $\text{Com}^\circ$ were constructed using different methods.

**8.1.2. Homotopy Invariance of Homotopy Bialgebras.** Suppose $G = \text{Gr}^!_c$ is the one-colored pasting scheme of connected wheel-free graphs, so $G$-props in $M$ are one-colored chain properads. Being $G_0$-cofibrant means that the unit map

$$\gamma^! : \mathbb{I} \longrightarrow P^!(\cdot)$$

is an injection. Suppose $B$ is the properad whose algebras [YJ15] (Def. 13.17) are biassociative bialgebras [Mar08] (Examples 59 and 62), which will simply be called bialgebras from now on. The properad $B$ is $G_0$-cofibrant, i.e., well-pointed and $\Sigma$-cofibrant.

By Theorem 7.3.2 the augmentation

$$B_\infty := W(\mathcal{G}, J, B) \xrightarrow{\eta} B$$

is a cofibrant resolution of the properad $B$. Since $B$-algebras are bialgebras, $B_\infty$-algebras can legitimately be called $A_\infty$-bialgebras. Algebras over $B_\infty$ are homotopy invariant in the following sense.

**Theorem 8.1.3.** Suppose $f : X \longrightarrow Y$ is a map in $M$. Then the following statements hold.

1. Suppose $f$ is an acyclic cofibration (i.e., an injective homology isomorphism). Then every $B_\infty$-algebra structure on $X$ naturally induces one on $Y$ such that $f$ is a map of $B_\infty$-algebras.
(2) Suppose $f$ is an acyclic fibration (i.e., a surjective homology isomorphism). Then every $B_\infty$-algebra structure on $Y$ naturally induces one on $X$ such that $f$ is a map of $B_\infty$-algebras.

**Proof.** This is a special case of (the proof of) [JY09] Theorem 1.2, which deals with algebras over a cofibrant $\mathcal{C}$-colored prop in a cofibrantly generated monoidal model category in which the colored operad for $\mathcal{C}$-colored props is admissible. Since $B_\infty$ is a cofibrant properad, that proof can be used verbatim for $B_\infty$-algebras. □

**Remark 8.1.4.** In [Mar06] a different resolution of $B_\infty$, called a minimal model, was constructed. The authors do not know how $B_\infty$-algebras are related to the $A_\infty$-bialgebras in [SU05].

### 8.2. Rectification and Homotopy Invariance of Homotopy Algebras

Throughout this section, suppose $\mathcal{M}$ is a cofibrantly generated monoidal model category. Recall that an operad $P$ in $\mathcal{M}$ is said to be admissible if the category of $P$-algebras admits a model category structure with weak equivalences and fibrations defined entrywise in $\mathcal{M}$. For example, by [WY17] (Section 8) every colored operad is admissible if $\mathcal{M}$ is the model category of (pointed or unpointed) simplicial sets [Quil67], of (bounded or unbounded) chain complexes or simplicial modules over a field of characteristic 0 [Hov99], of symmetric spectra with the positive (flat) stable model structure [Shi04], or of the category of small categories with the folk model structure [Rez].

In this section, we will work with the pasting scheme $\mathcal{G} = \text{UTree}$ of $\mathcal{C}$-colored unital trees, so $\mathcal{G}$-props in $\mathcal{M}$ are $\mathcal{C}$-colored operads. For an operad $P$ and a fixed commutative interval $J$ in $\mathcal{M}$, we will abbreviate $W(\text{UTree}, J, P)$ to $WP$.

#### 8.2.1. Rectification

**Motivation 8.2.1.** For a sufficiently nice operad $P$, its $W$-construction provides a cofibrant resolution of $P$ by Theorem 7.3.2. Since the augmentation $\eta : WP \to P$ is a weak equivalence and $WP$ is a cofibrant operad, it is reasonable to expect that $P$-algebras and $WP$-algebras are closely related. To make this precise, first observe that the augmentation induces a pre-composition functor

$$\eta^* : \text{Alg}(P) \to \text{Alg}(WP)$$

in which $\text{Alg}(P)$ denotes the category of $P$-algebras in $\mathcal{M}$. By the Adjoint Lifting Theorem ([Bor94] Theorem 4.5.6), the previous functor is the right adjoint of an adjunction

$$\begin{array}{ccc}
\text{Alg}(WP) & \xrightarrow{\eta} & \text{Alg}(P) \\
\eta^* & \downarrow & \\
& & 
\end{array}$$

The next result says that, under suitable conditions, this adjunction induces an equivalence of homotopy categories.

**Theorem 8.2.3.** Suppose $\mathcal{M}$ has a commutative interval in which the operad for $\mathcal{C}$-colored operads is admissible. Suppose $P$ is a $\Sigma_\ast$-cofibrant $\mathcal{C}$-colored operad in $\mathcal{M}$ such that both $P$ and $WP$ are admissible. Then the adjunction $(\eta, \eta^*)$ in $(8.2.2)$ is a Quillen equivalence, provided either one of the following two conditions holds.

1. $\mathcal{M}$ is a left proper model category and has a cofibrant monoidal unit.
(2) Every generating cofibration in \( M \) has a cofibrant domain.

In particular, when either (1) or (2) holds, then for each \( WP \)-algebra \( X \), there exist a \( P \)-algebra \( Y \) and a weak equivalence

\[ f : X \xrightarrow{\sim} \eta^* Y \]

of \( WP \)-algebras.

**Proof.** By Theorem 7.3.2 the augmentation

\[ \eta : WP \xrightarrow{\sim} P \]

is a weak equivalence with \( WP \) a cofibrant operad. Since the \( W \)-construction \( WP \) is cofibrant, it is also \( \Sigma \)-cofibrant. This fact is [BM03] (Prop. 4.3) in the one-colored case and [WY17] (Theorem 6.2.3(2) for the operad of \( \mathcal{C} \)-colored operads) for the general colored case. Moreover, in the proof of Theorem 7.2.17 we showed that each entry map

\[ WP_n(\ell) \xrightarrow{\sim} WP_{n+1}(\ell) \]

is an acyclic cofibration in \( M \). Therefore, so is their countable composition

\[ P(\ell) \xrightarrow{\sim} WP_0(\ell) \]

by [Hir03] (10.3.4). Since each colored unit of \( P \) is a cofibration by assumption, the same is also true for \( WP \). So the augmentation \( \eta \) is a weak equivalence between admissible \( \Sigma^* \)-cofibrant operads.

Under assumption (1), the adjunction \((\eta, \eta^*)\) in (8.2.2) is a Quillen equivalence by [BM07] (Theorem 4.1). Under assumption (2), the same holds by [WY∞] (Theorem 4.3.2). For the last assertion, when either (1) or (2) holds, simply apply the total left derived functor of the left Quillen equivalence \( \eta \) to a \( WP \)-algebra. \( \square \)

**Remark 8.2.4.** Case (1) of Theorem 8.2.3 is [BM07] Cor. 5.1.

**8.2.2. Homotopy Invariance.**

**Motivation 8.2.5.** For an operad \( P \), a \( P \)-algebra is an operad map from \( P \) to the endomorphism operad of an object. As a general principle, maps out of a cofibrant object are better behaved than general maps. So if the \( W \)-construction is a cofibrant resolution of \( P \), then \( WP \)-algebras should have nice homotopical properties. The following result is one way to make this precise and demonstrates another utility of a cofibrant resolution of an operad.

**Theorem 8.2.6.** Suppose \( M \) has a commutative interval and a cofibrant monoidal unit in which the operad for \( \mathcal{C} \)-colored operads is admissible. Suppose \( P \) is a \( \Sigma^* \)-cofibrant \( \mathcal{C} \)-colored operad in \( M \). Suppose

\[ f = \{ f_c : X_c \longrightarrow Y_c \}_{c \in \mathcal{C}} \]

is a collection of maps in \( M \). Then the following statements hold.

(1) Suppose each object \( Y_c \) with \( c \in \mathcal{C} \) is fibrant and that all of the maps \( \bigotimes_{i=1}^n f_{c_i} \) with \( n \geq 0 \) and each \( c_j \in \mathcal{C} \) are acyclic cofibrations. Then every \( WP \)-algebra structure on \( X = \{ X_c \} \) induces one on \( Y = \{ Y_c \} \) such that \( f \) is a map of \( WP \)-algebras.
8.3. Rectification of Homotopy Coherent Diagrams

Suppose a commutative segment $J$ in the underlying monoidal model category $\mathcal{M}$ has been fixed. In this section, we discuss rectification of homotopy coherent diagrams in $\mathcal{M}$ to strictly commutative diagrams. When the underlying category is $\text{Top}$, rectifying homotopy coherent diagrams to strictly commutative diagrams has a long history; see, e.g., [CP86, DKW89, Vog73]. In [BM03b] rectification of homotopy coherent diagrams in $\text{Top}$ is used to provide a conceptual framework for higher homotopy operations. A modern treatment of rectification of homotopy coherent diagrams in $\text{Top}$ is [BM07] Section 5. In this section, we will observe
that, under suitable conditions, homotopy coherent diagrams in $M$ can always be rectified.

8.3.1. Strictly Commutative Diagrams. Throughout the rest of this section, we will use the fact that, for any set $C$, the following three classes of objects in $M$ are the same:

1. $ULin$-props, where $ULin$ is the $C$-colored pasting scheme of unital linear graphs;
2. $M$-enriched categories with object set $C$;
3. $C$-colored operads $P$ with only unary operations, i.e., $P(c) = \emptyset$ if $|c| \neq 1$.

Recall that, for a small category $C$ with object set $C$, a $C$-diagram in $M$ is a functor $C \to M$.

Unraveling the definitions one obtains the following operadic reformulation of a $C$-diagram.

**Proposition 8.3.1.** For a small category $C$ with object set $C$, $C$-diagrams in $M$ are precisely algebras over the $C$-colored operad $C_{\text{diag}}$ in $M$ with entries

\[
C_{\text{diag}}(c) = \begin{cases} 
\emptyset & \text{if } |c| \neq 1; \\
\bigoplus_{f \in C(c,d)} \mathbb{1}_f & \text{if } c = e \in C,
\end{cases}
\]

where $\mathbb{1}_f$ is a copy of the monoidal unit in $M$. Its colored units correspond to the identity maps in $C$, and its operadic composition corresponds to the categorical composition in $C$.

**Example 8.3.3.** Consider the category $C$ with four objects $\{a, b, c, d\}$, five non-identity maps, and one commutative diagram, as in

```
\begin{array}{ccc}
a & \xrightarrow{f} & b \\
\downarrow^{e} & & \downarrow^{g} \\
c & \xrightarrow{h} & d
\end{array}
```

where the diagonal map $gf = he$ is not displayed. A $C$-diagram $X$ in $M$ is a commutative square. The $\{a, b, c, d\}$-colored operad $C_{\text{diag}}$ in $M$ has entries

\[
C_{\text{diag}}(x) = \begin{cases} 
\mathbb{1} & \text{if } (x) = (e), (f), (g), (h), (i) \text{ or } (\epsilon) \text{ for } x \in \{a, b, c, d\}; \\
\emptyset & \text{otherwise.}
\end{cases}
\]

As explained in [BM03b] Example 3.12, rectification of homotopy coherent $C$-diagrams in $\text{Top}$, which we will defined below, is closely related to the Toda bracket of three composable maps

\[
A \xrightarrow{p} B \xrightarrow{q} C \xrightarrow{r} D
\]

with both composites $qp$ and $rq$ null-homotopic.

**Example 8.3.4.** Consider the category $D$ consisting of the following commutative diagram

```
\begin{array}{cccccccc}
A_n & \longrightarrow & A_{n-1} & \longrightarrow & \cdots & \longrightarrow & A_1 & \longrightarrow & A_0 \\
\downarrow & & \downarrow & & \cdots & & \downarrow & \\
B_n & \longrightarrow & B_{n-1} & \longrightarrow & \cdots & \longrightarrow & B_1 & \longrightarrow & B_0
\end{array}
```

8.3. Rectification of Homotopy Coherent Diagrams

with \( n \geq 2 \). A D-diagram in \( \mathcal{M} \) is a commutative diagram in \( \mathcal{M} \) with the above shape. The \( \{ A_i, B_j \}_{i,j=0}^n \)-colored operad \( \mathcal{D}^{\text{diag}} \) in \( \mathcal{M} \) has entries

\[
\mathcal{D}^{\text{diag}}(\ell) = \begin{cases} 1 & \text{if } (\ell) = (i_j, (n_i)) \text{, or } (A_i, (n_i)) \text{, or } (B_i, (n_i)) \text{ for some } 1 \leq i \leq j \leq n; \\ \emptyset & \text{otherwise.} \end{cases}
\]

Homotopy commutative D-diagrams are called \( n \)-box diagrams in \([\text{MO1}1]\) and are used to define long box bracket operations.

8.3.2. Homotopy Coherent Diagrams.

Motivation 8.3.5. We proved in Theorem 7.3.2 that for a connected pasting scheme \( \mathcal{G} \) and a nice enough \( \mathcal{G} \)-prop \( \mathcal{P} \), the augmentation \( \eta : W(\mathcal{G}, J, P) \rightarrow P \) from the \( W \)-construction is a cofibrant resolution of \( P \). It is therefore natural to regard \( W(\mathcal{G}, J, P) \)-algebras as homotopy \( \mathcal{P} \)-algebras. If \( \mathcal{P} \) is the operad \( \mathcal{C}^{\text{diag}} \) for \( \mathcal{C} \)-diagrams in \( \mathcal{M} \), then algebras over its \( W \)-construction can rightfully be regarded as homotopy \( \mathcal{C} \)-diagrams.

Definition 8.3.6. Suppose \( \mathcal{C} \) is a small category with object set \( \mathcal{C} \).

1. Define the \( \text{ULin} \)-prop in \( \mathcal{M} \) (Theorem 3.5.17)

\[
W^{\text{diag}} = \text{W}\left( \text{ULin}, J, \mathcal{C}^{\text{diag}} \right),
\]

where \( \mathcal{C}^{\text{diag}} \) is the \( \text{ULin} \)-prop in Prop. 8.3.1.

2. A homotopy coherent \( \mathcal{C} \)-diagram in \( \mathcal{M} \) is a \( W^{\text{diag}} \)-algebra, where \( W^{\text{diag}} \) is regarded as a \( \mathcal{C} \)-colored operad with only unary operations.

After the Rectification Theorem below, we will unwrap this definition and explain what a \( W^{\text{diag}} \)-algebra is explicitly. In particular, we will see why it makes sense to call these objects homotopy coherent \( \mathcal{C} \)-diagrams.

Motivation 8.3.7. The augmentation

\[
\eta : W^{\text{diag}} \rightarrow \mathcal{C}^{\text{diag}}
\]

in Prop. 4.1.2 induces an adjunction

\[
\{ \text{h.c. \( \mathcal{C} \)-diagrams} \} = \text{Alg}(W^{\text{diag}}) \xleftarrow{\eta^*} \text{Alg}(\mathcal{C}^{\text{diag}}) = \{ \mathcal{C} \text{-diagrams} \}
\]

as in (8.2.2), where the right adjoint \( \eta^* \) is given by pre-composition with \( \eta \). The next result is the special case of Theorem 8.2.3 for \( \mathcal{C} \)-colored operads with only unary operations (i.e., \( \text{ULin}^{\mathcal{C}} \)-props, or equivalently \( \mathcal{M} \)-enriched categories with object set \( \mathcal{C} \)), so it requires no further proof. It ensures that homotopy coherent \( \mathcal{C} \)-diagrams can be rectified to \( \mathcal{C} \)-diagrams.

Theorem 8.3.8. Suppose the operad for \( \text{ULin} \)-props is admissible in \( \mathcal{M} \). Suppose \( \mathcal{C} \) is a small category with object set \( \mathcal{C} \) such that:

- Each colored unit of \( \mathcal{C}^{\text{diag}} \) is a cofibration.
- Both \( \mathcal{C} \)-colored operads \( \mathcal{C}^{\text{diag}} \) and \( W^{\text{diag}} \) are admissible in \( \mathcal{M} \).

Then the adjunction

\[
\text{Alg}(W^{\text{diag}}) \xleftarrow{\eta^*} \text{Alg}(\mathcal{C}^{\text{diag}})
\]

is a Quillen equivalence, provided either one of the following two conditions holds.

1. \( \mathcal{M} \) is a left proper model category and has a cofibrant monoidal unit.
(2) Every generating cofibration in \( M \) has a cofibrant domain.
In particular, when either (1) or (2) holds, then for each homotopy coherent \( C \)-
diagram \( X \) in \( M \), there exist a \( C \)-diagram \( Y \) in \( M \) and a weak equivalence
\[ f : X \xrightarrow{\sim} \eta^* Y \]
of \( WC^{\text{diag}} \)-algebras.

8.3.3. Unraveling Homotopy Coherent Diagrams. Suppose \( C \) is a small
category with object set \( \mathcal{C} \). Here we provide a more explicit description of \( WC^{\text{diag}} \)-
algebras, i.e., homotopy coherent \( C \)-diagrams in \( M \).

A typical entry of the \( U \text{Lin-prop} \) \( WC^{\text{diag}} \) is defined as a coend [3.4.3]
\[ WC^{\text{diag}}(c) = \int^L \mathcal{U} \text{Lin}^\mathcal{C}(c) \mathcal{J} \otimes C^{\text{diag}}[L] \]
for \( c, d \in \mathcal{C} \). If \( L \) is the \( c \)-colored exceptional edge \( \uparrow_c \) for some \( c \in \mathcal{C} \), then
\[ J[\uparrow_c] \otimes C^{\text{diag}}[\uparrow_c] = 1. \]
Suppose \( L \in \mathcal{U} \text{Lin}(\mathcal{C}) \) is the linear graph
\[ L(c_0, \ldots, c_n) = \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
\ldots \\
(n-1) \\
n
\end{array} \]
with \( n \geq 1 \) vertices, \( n-1 \) internal edges, and each \( c_i \in \mathcal{C} \). Then
\[ J[L] = J^{\otimes n-1} = J_{c_1} \otimes \cdots \otimes J_{c_{n-1}}, \]
where each \( J_{c_i} \) is a copy of the commutative interval \( J \). From the definition of the
operad \( C^{\text{diag}} \) [8.3.2], we have that
\[ C^{\text{diag}}[L] = \bigotimes_{i=1}^n C^{\text{diag}}(c_i) = \prod_{\{f_i\} \in \prod_{i \in \mathcal{C}(c_{i-1}, c_i)} (f_i)} \mathbb{1}(f_i), \]
where \( \mathbb{1}(f_i) \) is a copy of the monoidal unit \( \mathbb{1} \) indexed by the sequence
\[ c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots c_{n-1} \xrightarrow{f_n} c_n \]
of \( n \) composable maps in \( \mathcal{C} \). So we have that
\[ (8.3.9) \quad J[L] \otimes C^{\text{diag}}[L] \cong \prod_{\{f_i\} \in \prod_{i \in \mathcal{C}(c_{i-1}, c_i)} (f_i)} J_{c_1} \otimes \cdots \otimes J_{c_{n-1}}. \]

Suppose \( X \) is a \( WC^{\text{diag}} \)-algebra in \( M \). Since \( WC^{\text{diag}} \) is a \( \mathcal{C} \)-colored operad with
only unary operations, this means \( X = \{ X_e \in M \}_{e \in \mathcal{C}} \) is equipped with \( WC^{\text{diag}} \)-algebra
structure maps
\[ WC^{\text{diag}}(c) \otimes X_c \xrightarrow{\lambda} X_d \]
for \( c, d \in \mathcal{C} \) that satisfy the \( WC^{\text{diag}} \)-algebra unity and associativity axioms. Using
[8.3.9] and Lemma [3.4.4], this means that for each sequence \( (c_0, \ldots, c_n) \in \mathcal{C}^{n+1} \)
with \( n \geq 1 \) and each sequence of composable maps \( \{f_i\}_{i=1}^n \in \prod_{i=1}^n \mathcal{C}(c_{i-1}, c_i) \), it is
equipped with a structure map
\[ (8.3.10) \quad J_{c_1} \otimes \cdots \otimes J_{c_{n-1}} \otimes X_{c_0} \xrightarrow{X(f_1, \ldots, f_n)} X_{c_n} \]
\[ \xrightarrow{(\text{inclusion,Id})} \]
\[ J[L] \otimes C^{\text{diag}}[L] \otimes X_c \xrightarrow{(\omega L, \text{Id})} WC^{\text{diag}}(c) \otimes X_c \]
in \( M \) that satisfies the following axioms.

**Unity:** If \( n = 1 \) then
\[
X(Id_c) = Id_{X_c}
\]
for each \( c \in C \). Moreover, if \( n \geq 2 \) and \( f_i = Id_{c_i} \) (so \( c_{i-1} = c_i \)) for some \( 1 \leq i \leq n \), then the diagram
\[
\begin{array}{ccc}
J_{c_1} \otimes \cdots \otimes J_{c_{n-1}} \otimes X_{c_0} & \xrightarrow{X(f_1, \ldots, Id_{c_i}, \ldots, f_n)} & X_{c_n} \\
J \otimes^{n-2} X_{c_0} & \xrightarrow{X(f_1, \ldots, Id_{c_i}, \ldots, f_n)} & X_{c_n}
\end{array}
\]
is commutative, where in the bottom horizontal map \( Id_{c_i} \) is omitted. The left vertical map is the identity map tensored with
- the counit \( \epsilon : J_{c_1} \longrightarrow 1 \) if \( i = 1 \);
- the multiplication \( J_{c_{i-1}} \otimes J_{c_i} \longrightarrow J \) if \( 1 < i < n \);
- the counit \( \epsilon : J_{c_{n-1}} \longrightarrow 1 \) if \( i = n \).

**Composition:** If \( n \geq 2 \) and \( 1 \leq i \leq n-1 \), then the diagram
\[
(8.3.11)
\]
is commutative, where the top horizontal map is the identity map tensored with \( 0 : 1_{c_{i}} \longrightarrow J_{c_{i}} \). In the bottom horizontal map, \( f_i \) and \( f_{i+1} \) have been composed.

**Associativity:** Given composable maps
\[
c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n+p}} c_{n+p}
\]
in \( C \) with \( n, p \geq 1 \), the diagram
\[
(8.3.12)
\]
is commutative, where the lower left vertical map is the identity map tensored with \( 1 : 1_{c_n} \longrightarrow J_{c_n} \).

In particular, in \( 8.3.10 \) with \( n = 1 \), the \( W \mathbb{C}^{\delta\delta} \)-algebra \( X \) is equipped with a structure map
\[
X_c \xrightarrow{X(f)} X_d
\]
in \( M \) for each map \( f \in C(c,d) \), just like a \( C \)-diagram. For \( n \geq 2 \) the structure maps \( X(f_1, \ldots, f_n) \) provide a system of coherent higher homotopies in the following sense.
Given two composable maps
\[ c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \]
in \( C \), the structure map
\[ J_{c_2} \otimes X_{c_0} \xrightarrow{X(f_1, f_2)} X_{c_2} \]
is a specific homotopy from \( X(f_2 f_1) \) to the composite \( X(f_2) X(f_1) \) in the sense that there is a commutative diagram
\[
\begin{array}{ccc}
1 \otimes X_{c_0} & \xrightarrow{\cong} & X_{c_0} \\
\downarrow & & \downarrow \\
J \otimes X_{c_0} & \xrightarrow{X(f_1, f_2)} & X_{c_2} \\
\downarrow & & \downarrow \\
1 \otimes X_{c_0} & \xrightarrow{\cong} & X_{c_0}
\end{array}
\]
in \( M \). The top square is the composition axiom \((8.3.11)\) with \( n = 2 \), and the bottom square is the associativity axiom \((8.3.12)\) with \( n = p = 1 \). Thinking of it as a homotopy, we may depict \( X(f_1, f_2) \) as
\[
X(f_2 f_1) \xrightarrow{X(f_1, f_2)} X(f_2) X(f_1)
\]
which is not a map in \( M \).

Likewise, given three composable maps
\[ c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} c_3 \]
in \( C \), there are commutative diagrams
\[
\begin{array}{ccc}
1_{c_1} \otimes J_{c_2} \otimes X_{c_0} & \xrightarrow{\cong} & J_{c_2} \otimes X_{c_0} \\
\downarrow (0,1d) & & \downarrow \\
J_{c_1} \otimes J_{c_2} \otimes X_{c_0} & \xrightarrow{X(f_1, f_2, f_3)} & X_{c_3} \\
\downarrow (1,1d) & & \downarrow \\
1_{c_1} \otimes J_{c_2} \otimes X_{c_0} & \xrightarrow{\cong} & J_{c_2} \otimes X_{c_1} \\
\downarrow & & \downarrow \\
J_{c_2} \otimes X_{c_0} & \xrightarrow{X(f_1, f_2)} & X_{c_2}
\end{array}
\]
in \( M \). In the left (resp., right) diagram, the top square is the composition axiom \((8.3.11)\) with \( n = 3 \) and \( i = 1 \) (resp., \( i = 2 \)), and the bottom square is the associativity axiom \((8.3.12)\) with \( n = 1 \) and \( p = 2 \) (resp., \( n = 2 \) and \( p = 1 \)). Thinking of it as a
homotopy, we may depict \( X(f_1, f_2, f_3) \) as a filled-in square

\[
\begin{array}{c@{\quad}c@{\quad}c}
X(f_3)X(f_2f_1) & X(f_3)X(f_2)X(f_1) \\
X(f_2f_1f_3) & (f_3 \otimes X(f_1))X(f_2f_1) \\
X(f_3f_2f_1) & X(f_1)X(f_3f_2) \\
\end{array}
\]

which is again not a diagram in \( \mathcal{M} \).

There are similar commutative diagrams for the structure maps \( X(f_1, \ldots, f_n) \) with \( n \geq 4 \). In summary, in a \( \mathcal{C} \)-diagram \( Y \) in \( \mathcal{M} \), given any sequence of composable maps \( \{ f_i \} \in \prod_{i=1}^n \mathcal{C}(c_{i-1}, c_i) \) with \( n \geq 2 \), there is a uniquely defined composite

\[
Y(f_n) \cdots Y(f_1) = Y(f_n \cdots f_1) : Y_{c_0} \longrightarrow Y_{c_n}.
\]

In a homotopy coherent \( \mathcal{C} \)-diagram, these equalities are replaced by a system of coherent higher homotopies. Similar discussion can be found in [BM07, BM03b, CP86, Vog73]. The \( W \)-construction \( WC^{\text{diag}} \) in this case neatly packages all the higher homotopy coherence information.

### 8.4. Graphically Generated Generalized Props

Fix a 1-colored pasting scheme \( \mathcal{G} \). In this section, we apply Theorem 7.3.2 and observe that the Boardman-Vogt construction provides a cofibrant resolution of \( \mathcal{G} \)-props generated by graphs. When restricted to various pasting schemes, these graphically generated \( \mathcal{G} \)-props are important in the study of \( \infty \)-operads [MT10], \( \infty \)-properads, and \( \infty \)-wheeled properads [HRY15]. Also fix a commutative segment \( (J, \mu, 0, 1, \epsilon) \) in our ambient category \( (\mathcal{M}, \otimes, \mathbb{I}) \), which is cocomplete in which the monoidal product commutes with colimits on both sides. Since the pasting scheme \( \mathcal{G} \) and \( J \) are understood, we will omit them from the notation of the \( W \)-construction, so

\[
WP = W(\mathcal{G}, J, P).
\]

#### 8.4.1. Defining Graphically Generated Generalized Props

We first define the \( \mathcal{G} \)-props generated by graphs.

**Definition 8.4.1.** If \( \mathcal{C} \) is a set, we also consider the \( \mathcal{C} \)-colored version of the 1-colored pasting scheme \( \mathcal{G} \), denoted \( \mathcal{G}_\mathcal{C} \), in which graphs are now equipped with a \( \mathcal{C} \)-coloring on edges. The category of \( \mathcal{G}_\mathcal{C} \)-props in \( \mathcal{M} \) is denoted by \( \text{Prop}^\mathcal{G}_\mathcal{C}(\mathcal{M}) \).

**Definition 8.4.2.** Suppose \( G \in \mathcal{G} \).

1. Equipping edges in \( G \) with the identity \( \text{Ed}(G) \)-coloring (i.e., each edge is \( \text{Ed}(G) \)-colored by itself), each vertex \( v \) in \( G \) and the full graph \( G \) have associated \( \text{Ed}(G) \)-profiles.

2. Define \( \widetilde{G} \) as the \( \text{Prof}(\text{Ed}(G))^{x^2} \)-graded set with

\[
\widetilde{G}(\underline{v}) = \begin{cases} 
\{ v \} & \text{if } \underline{v} = (\nu(v)) \text{ for some } v \in \text{Vt}(G), \\
\emptyset & \text{otherwise},
\end{cases}
\]

for \( \underline{v} \in \text{Prof}(\text{Ed}(G))^{x^2} \).
(3) Define
\[ \Gamma^G \in \text{Prop}_{\text{Ed}(G)}(\text{Set}) \]
as the \( \text{Ed}(G) \)-colored \( G \)-prop in \( \text{Set} \) freely generated by \( \tilde{G} \).

(4) Define
\[ \Gamma^G_M \in \text{Prop}_{\text{Ed}(G)}(M) \]
as the image of \( \Gamma^G \) under the strong symmetric monoidal functor \( \text{Set} \to M \) that sends \( T \) to \( \coprod_T 1 \).

Remark 8.4.3. When the pasting scheme \( G \) is simply connected (e.g., those for small categories, operads, and dioperads), each entry of \( \Gamma^G \) is either empty or a singleton, and \( \Gamma^G \) is a finite set. Moreover, a map between two such graphically generated \( G \)-props is uniquely determined by its action between edge sets [MT10] (Section 2.3). The situation for non-simply connected pasting schemes is far more complicated, as explained in [HRY15] (Chapter 5). In fact, when \( G \) is a non-simply connected pasting scheme, \( \Gamma^G \) is a finite set precisely when \( G \) is a simply connected graph [HRY15] (Theorem 5.9). Furthermore, a \( G \)-prop map between two such graphically generated \( G \)-props is in general not determined by its action between edge sets [HRY15] (Section 5.3.2).

Example 8.4.4. Consider the 1-colored pasting scheme \( \text{Gr}_{\text{ lcirclearrowdown}} \) of connected wheeled graphs and the graph \( G \) with one vertex \( v \) and one directed loop \( a \) at \( v \).

So \( \Gamma^G \) is the \( \{a\} \)-colored wheeled properad in \( \text{Set} \) freely generated by \( v \in \text{Vt}(G) \) with \( (\text{out}(v)) = (\emptyset) \). Let use describe the entries of \( \Gamma^G \).

(1) The entry \( \Gamma^G(\emptyset) \) consists of the \( \{a\} \)-colored linear graphs
\[ L_n = \]
for \( n \geq 0 \), in which there are \( n \) copies of the vertex \( v \). Note that \( L_0 \) is the exceptional edge \( a \). The wheeled properad structure restricted to the entry \( \Gamma^G(\emptyset) \) yields the free monoid on one object \( v \) with multiplication corresponding to grafting of linear graphs.

(2) The entry \( \Gamma^G(\{a\}) \) consists of the \( \{a\} \)-colored directed cycles
\[ \xi_1 L_n = \]
for \( n \geq 0 \), in which there are \( n \) copies of the vertex \( v \). Note that \( \xi_1 L_0 \) is the exceptional loop \( \{\ } \). The contraction
\[ \xi_1 : \Gamma^G(\emptyset) \to \Gamma^G(\{a\}) \]
sends \( L_n \) to \( \xi_1 L_n \).

All other entries of \( \Gamma^G \) are empty.
Example 8.4.5. Consider the same graph $G$ as in Example 8.4.4 but with the pasting scheme $\text{Gr}^G$ of wheeled graphs. So here $\Gamma^G$ is the $\{a\}$-colored wheeled prop in $\text{Set}$ freely generated by $v \in Vt(G)$ with $\left(\omega^{\text{out}}(v)\right) = \left(\omega^a\right)$. More concretely, $\Gamma^G$ consists of wheeled graphs in which each connected component is a connected wheeled graph $L_n$ or $\xi_1^1L_n$ as in Example 8.4.4. This is true because the pair

$\text{Gr}^G \leq \text{Gr}^G$

of pasting schemes is well-matched; see [YJ15] Lemma 9.11, Lemma 9.19, and Lemma 12.3.

8.4.2. Cofibrant Resolution of Graphically Generated Generalized Props.

Corollary 8.4.6. Suppose $G$, $J$, and $M$ are as in Theorem 7.3.2, and $G \in G$. Then the augmentation

$$W^G_M \xrightarrow{\eta} \Gamma^G_M$$

is a cofibrant resolution of the $\text{Ed}(G)$-colored $G$-prop $\Gamma^G_M$ in $M$.

Proof. Using Theorem 7.3.2 we need to show $\Gamma^G_M$ is $G_0$-cofibrant; see Remark 7.2.13. Since $1$ is cofibrant, both maps

$$1 \xrightarrow{\eta}\Gamma^G_M(\cdot) = \bigsqcup_{r \in \text{Ed}(G)} 1 \quad \text{and} \quad 1 \xrightarrow{\eta}\Gamma^G_M(\cdot) = \bigsqcup_{r \in \text{Ed}(G)} 1$$

are cofibrations for each $c \in \text{Ed}(G)$. To show that $\Gamma^G_M$ is $\Sigma$-cofibrant, it suffices to observe that $\Gamma^G$ is $\Sigma$-cofibrant, which is true because the $\Sigma$-action permutes the graph listings of the graphs in the freely generated $G$-prop $\Gamma^G$. □
CHAPTER 9

Bar Resolution

We continue to assume \((\mathcal{M}, \emptyset, 1)\) is a cocomplete symmetric monoidal category in which the monoidal product commutes with colimits on both sides, which is automatic if \(\mathcal{M}\) is also closed. Throughout this chapter, \(\mathcal{G}\) is a fixed \(\mathcal{C}\)-colored pasting scheme. Recall that \(\mathcal{G}_0\) is the sub-pasting scheme of graphs with no ordinary internal edges. The main result of this chapter is Theorem 9.5.5. It provides an explicit identification between the following two kinds of resolutions of \(\mathcal{G}\)-props:

1. The bar resolution corresponding to the free-forgetful adjunction
   \[
   \text{Prop}^{\mathcal{G}_0}(\mathcal{M}) \xrightarrow{F^\mathcal{G}} \text{Prop}^\mathcal{G}(\mathcal{M}) \xleftarrow{U}.
   \]

2. An instance of the Boardman-Vogt resolution in simplicial objects in \(\mathcal{M}\).

As a consequence of this identification, a suitable realization of the bar resolution is an instance of the Boardman-Vogt construction in \(\mathcal{M}\); see Theorem 9.6.4.

For technical reasons that we will explain in the course of the proof, our identification of the bar resolution as an instance of the \(W\)-construction works when the pasting scheme \(\mathcal{G}\) is shrinkable (Def. 9.2.5). This includes unital linear graphs (for small categories), unital trees (for operads), simply-connected graphs (for di-operads), wheeled trees (for wheeled operads), and connected wheeled graphs (for wheeled properads). However, the pasting schemes of all wheeled graphs and of (connected) wheel-free graphs are not shrinkable, so our identification does not cover prop(erad)s and wheeled props.

In Section 9.1 we discuss the bar resolution of a \(\mathcal{G}\)-prop with respect to an inclusion of pasting schemes. In Section 9.2 we observe that the free functor applied to the underlying \(\mathcal{G}_0\)-prop of the \(W\)-construction of a \(\mathcal{G}\)-prop is again a \(W\)-construction but with a different commutative segment. In Section 9.3 we show that each level of the bar resolution of a \(\mathcal{G}\)-prop is a \(W\)-construction. In Section 9.4 we discuss the \(W\)-construction when the ambient category is the category of simplicial objects in \(\mathcal{M}\). In Section 9.5 we prove that the bar resolution of a \(\mathcal{G}\)-prop is naturally isomorphic to the \(W\)-construction in simplicial objects in \(\mathcal{M}\). In Section 9.6 we observe that a suitable realization of the bar resolution of a \(\mathcal{G}\)-prop is again the \(W\)-construction in \(\mathcal{M}\).

9.1. Bar Resolution for a Pair of Pasting Schemes

Let us first spell out the details of the bar resolution for \(\mathcal{G}\)-props for an arbitrary \(\mathcal{C}\)-colored pasting scheme \(\mathcal{G}\) relative to the sub-pasting scheme \(\mathcal{G}_0\). Our description below actually works more generally for any pasting scheme inclusion, not just \(\mathcal{G}_0 \leq \mathcal{G}\). For the rest of this section, let us assume

\[
\mathcal{G} = (S, \mathcal{G}) \leq (S', \mathcal{G}') = \mathcal{G}'.
\]
is an inclusion of \( \mathcal{C} \)-colored pasting schemes. This pasting scheme inclusion yields a free-forgetful adjunction \((1.7.2)\)

\[
\text{Prop}^G(M) \xrightarrow{F^G, G'} U \xleftarrow{\varepsilon} \text{Prop}^{G'}(M)
\]

whose left adjoint is explicitly described in Lemma \(1.7.4\) using the extension category \(\mathcal{D}(\xi)\) in Def. \(1.7.3\).

Recall that the objects in \(\mathcal{D}(\xi)\) are the strict isomorphism classes of graphs in \(G(\xi)\) in which each vertex profile lies in \(S\). We will also write the object set of \(\mathcal{D}(\xi)\) as \(G_S(\xi)\).

**Example 9.1.1.** Suppose \( G \) is a \( \mathcal{C} \)-colored pasting scheme, and \( (\xi) \) is a pair of \( \mathcal{C} \)-profiles.

1. For the inclusion \( G_0 \leq G \) with \( G_0 \) as in Def. \(4.2.1\), the free-forgetful adjunction \((9.1.2)\)

\[
\text{Prop}^{G_0}(M) \xrightarrow{F^{G_0}} \text{Prop}^{G}(M) \xleftarrow{\varepsilon} \text{Prop}^{G'}(M)
\]

is our main example of interest.

2. For the identity inclusion \( G \leq G \), the extension category \(\mathcal{D}(\xi)\) is the substitution category \(G_S(\xi)\) in Def. \(3.1.2\).

As any adjunction does, the free-forgetful adjunction \(F^{G, G'} \rightarrow U\) \((1.7.2)\) induces a comonad on \(G'\)-props in \(M\) with:

- functor part \(F^{G, G'} U\);
- counit \(\varepsilon : F^{G, G'} U \xrightarrow{\varepsilon} \text{Id}\) that of the adjunction;
- comultiplication \(F^{G, G'} \nu U\), where \(\nu : \text{Id} \xrightarrow{\nu} UF^{G, G'}\) is the unit of the adjunction.

See, e.g., [Mac98] (VI.1 page 139). For each \(G'\)-prop \(P\) in \(M\), this comonad yields an augmented simplicial \(G'\)-prop

\[
(F^{G, G'} U)^{**} P \xrightarrow{\varepsilon} P
\]

in \(M\), called the **bar resolution** [Mac98] (VII.6) or the Godement resolution [God58] (App. 3) of \(P\). The augmentation \(\varepsilon\) is the counit of the adjunction \(F^{G, G'} \rightarrow U\). It is induced by the \(G'\)-prop structure map of \(P\) in the sense that the triangle

\[
P[K] \xrightarrow{\gamma^K} F^{G, G'} U P(\xi) \xrightarrow{\varepsilon} P(\xi)
\]

is commutative for each graph \(K\) in \(\mathcal{D}(\xi)\), i.e., in \(G_S(\xi)\).

To identify the bar resolution \((9.1.3)\) for the adjunction \(F^G \rightarrow U\) with an instance of the \(W\)-construction, first we need to understand the simplicial \(G'\)-prop \((F^{G, G'} U)^{**} P\) better. To understand its entries, we need the following bookkeeping device from [YJ15] (Def. 7.1) to keep track of iterated graph substitution data.
DEFINITION 9.1.5. Suppose \( \mathcal{G} \subseteq \mathcal{G}' \) is an inclusion of \( \mathcal{C} \)-colored pasting schemes. Suppose \( n \geq 0 \), and \( \frac{\mathcal{G}}{\mathcal{G}} \) is a pair of \( \mathcal{C} \)-profiles.

1. A graph \((n + 1)\)-simplex with profiles \( \frac{\mathcal{G}}{\mathcal{G}} \) is a sequence
\[
H = H^{[1, n + 1]} = (H^1, \ldots, H^{n+1})
\]
in which:
- \( H^{n+1} \in \mathcal{G}_S' \map{\mathcal{G}}{\mathcal{G}} \).
- For each \( 1 \leq j \leq n \), \( H^j \) is a finite family of graphs in \( \mathcal{G}_S' \) indexed by the set of vertices appearing in the family \( H^{j+1} \) such that each \( H^j \) has the same profiles as the indexing vertex \( v \in H^{j+1} \).

2. Given a graph \((n + 1)\)-simplex \( H \) as above, its graph substitution is defined as
\[
\text{sub}(H) = H^{n+1}(H^n) \cdots (H^1) \in \mathcal{G}_S' \map{\mathcal{G}}{\mathcal{G}}.
\]
which is obtained by performing all possible graph substitutions in \( H \).

3. A map of graph \((n + 1)\)-simplices is defined as
(9.1.6)
\[
(H^1(K^1), \ldots, H^{n+1}(K^{n+1})) \map{\frac{\mathcal{G}'}{\mathcal{G}}}{\frac{\mathcal{G}'}{\mathcal{G}}}(K^2(H^1), K^3(H^2), \ldots, K^{n+1}(H^n), H^{n+1}).
\]
Here each \( K^j \) is a finite family of graphs in \( \mathcal{G}_S' \) indexed by the set of vertices appearing in the family \( H^j \) such that each \( K^j \) has the same profiles as the indexing vertex \( v \in H^j \). Note that we are not asking \((K^1, \ldots, K^{n+1})\) to be a graph \((n + 1)\)-simplex.

4. Given composable maps \((K^1, \ldots, K^{n+1})\) followed by \((I^1, \ldots, I^{n+1})\) of graph \((n + 1)\)-simplices, their composite is defined by graph substitution as in:
\[
(H^1(I^1)(K^1), H^2(I^2)(K^2), \ldots, H^n(I^n)(K^n), H^{n+1}(I^{n+1})(K^{n+1})).
\]

5. With families of corollas as identities and graph substitution, which is unital and associative, as composition, there is a category of graph \((n + 1)\)-simplices with profiles \( \frac{\mathcal{G}}{\mathcal{G}} \) denoted by \( D^{n+1}(\frac{\mathcal{G}}{\mathcal{G}}) \).

EXAMPLE 9.1.7. Suppose \( \frac{\mathcal{G}}{\mathcal{G}} \) is a pair of \( \mathcal{C} \)-profiles.

1. A graph 1-simplex is a single graph. A map of graph 1-simplices has the form
\[
H(K) \map{\mathcal{K}}{\mathcal{H}} H
\]
with \( H \in \mathcal{G}_S' \map{\mathcal{G}}{\mathcal{G}} \) and each \( K_v \in \mathcal{G}_S(v) \) for \( v \in H \). So
\[
D^1(\frac{\mathcal{G}}{\mathcal{G}}) = D(\frac{\mathcal{G}}{\mathcal{G}}),
\]
the extension category in Def. 1.7.3

2. A graph 2-simplex \( H = (H^1, H^2) \) consists of:
• a graph \( H^2 \in \mathcal{G}_S(v) \);
• a family of graphs \( H^i_u \in \mathcal{G}_S(v) \) for \( v \in H^2 \).

A map of graph 2-simplices looks like

\[
\left( H^1(K^1), H^2(K^2) \right) \xrightarrow{(K^1,K^2)} \left( K^2(H^1), H^2 \right)
\]

with each \( K^j_i \in \mathcal{G}_S(v) \) for \( j = 1, 2 \) and \( v \in H^j \).

(3) A graph 3-simplex \( H = (H^1, H^2, H^3) \) consists of:
• a graph \( H^3 \in \mathcal{G}_S(v) \);
• a family of graphs \( H^2_v \in \mathcal{G}_S(v) \) for \( v \in H^3 \);
• a family of graphs \( H^1_u \in \mathcal{G}_S(u) \) for \( u \in H^2 \).

A map of graph 3-simplices looks like

(9.1.8) \(
\left( H^1(K^1), H^2(K^2), H^3(K^3) \right) \xrightarrow{(K^1,K^2,K^3)} \left( K^2(H^1), K^3(H^2), H^3 \right)
\)

with each \( K^j_i \in \mathcal{G}_S(v) \) for \( 1 \leq j \leq 3 \) and \( v \in H^j \).

**Definition 9.1.9.** For a pair \( \mathcal{G} \leq \mathcal{G}' \) of pasting schemes, suppose \( X \) is a \( \mathcal{G} \)-prop in \( M \).

1. Given a graph \((n+1)\)-simplex \( H = H^{[1,n+1]} \), define

\[
X[H] = X[\text{sub}(H)] = \bigotimes_{u \in \text{sub}(H)} X(u)
\]

where the unordered tensor product is indexed by the set of vertices in \( \text{sub}(H) \). Note that this makes sense because each vertex in \( \text{sub}(H) \) has profiles in \( S \).

2. Define the functor

\[
X : \mathcal{D}^{n+1}(\frac{\mathcal{G}}{\mathcal{G}'}) \to M
\]

whose value at \( H \in \mathcal{D}^{n+1}(\frac{\mathcal{G}}{\mathcal{G}'}) \) is \( X[H] \). For a map \( K = (K^1, \ldots, K^{n+1}) \) as in (9.1.6), define the map

(9.1.11) \[
X(K) = \bigotimes_{v \in H^1} \gamma_{K^1_v}^X : X[sK] \to X[tK].
\]

Here each

\[
\gamma_{K^1_v}^X : X[K^1_v] \to X(v)
\]

is a \( \mathcal{G} \)-prop structure map of \( X \), and \( sK \) and \( tK \) denote the source and the target of the map \( X(K) \).

**Remark 9.1.12.** For a \( \mathcal{G} \)-prop \( X \) and a graph \((n+1)\)-simplex \( H \), note that

\[
X[H] = X[\text{sub}(H)] = \begin{cases} X[H^1] & \text{if } n = 0 \\ \bigotimes_{v \in \text{sub}(H)} X[H^1] & \text{if } n \geq 1. \end{cases}
\]

**Example 9.1.13.** For a graph 3-simplex \( H = (H^1, H^2, H^3) \), the object \( X[H] \) is

\[
X[H] = \bigotimes_{u \in H^1} \bigotimes_{v \in H^2} X(u) = \bigotimes_{u \in H^1} X[H^1_u].
\]

For a map \( K \) of graph 3-simplices as in (9.1.8), the map \( X[K] \) is

\[
X[sK] = \bigotimes_{v \in G} X[K^1_v] \xrightarrow{\bigotimes \gamma_{K^1_v}^X} \bigotimes_{v \in G} X(v) = X[G] = X[tK]
\]
where
\[ G = \mathcal{H}^3(K1)(H2)(K2)(H1). \]

The following observation is about the entries of each level of the simplicial structure maps of the bar resolution.

**Theorem 9.1.14.** Suppose \( \mathcal{G} \leq \mathcal{G}' \) is an inclusion of \( \mathcal{C} \)-colored pasting schemes, and \( P \) is a \( \mathcal{G}' \)-prop in \( M \). For \( n \geq 0 \) and \( \frac{\bigtriangleup}{2} \) a pair of \( \mathcal{C} \)-profiles, there is a canonical isomorphism
\[
(F^{\mathcal{G}, \mathcal{G}'} U)^{n+1} P_{\frac{\bigtriangleup}{2}}(\frac{\bigtriangleup}{2}) \cong \colim_{\mathcal{H} \in \mathcal{D}^{n+1}(\frac{\bigtriangleup}{2})} \mathcal{P}[\mathcal{H}].
\]

**Proof.** This is proved by induction. The \( n = 0 \) case holds by Lemma 1.7.3 because \( \mathcal{D}^1(\frac{\bigtriangleup}{2}) \) is the extension category \( \mathcal{D}(\frac{\bigtriangleup}{2}) \). For \( n \geq 1 \) the induction step is proved by combining Lemma 1.7.3, the induction hypothesis, and the commutation of the monoidal product with colimits:
\[
(F^{\mathcal{G}, \mathcal{G}'} U)^{n+1} P_{\frac{\bigtriangleup}{2}}(\frac{\bigtriangleup}{2}) \cong \colim_{K \in \mathcal{D}(\frac{\bigtriangleup}{2})} \otimes (F^{\mathcal{G}, \mathcal{G}'} U)^n P(v)
\]
\[
\cong \colim_{K \in \mathcal{D}(\frac{\bigtriangleup}{2})} \otimes \colim_{v \in \mathcal{D}(v) \in \mathcal{D}^1(v)} \mathcal{P}[\mathcal{H}(v)]
\]
\[
\cong \colim_{K \in \mathcal{D}(\frac{\bigtriangleup}{2})} \colim_{v \in \mathcal{D}(v) \in \mathcal{D}^1(v)} \otimes \mathcal{P}[\mathcal{H}(v)].
\]

The last iterated colimit is isomorphic to the colimit in the statement. \( \Box \)

**Remark 9.1.15.** In the context of the previous Theorem, there is a natural map
\[
\mathcal{P}[\mathcal{H}] \longrightarrow (F^{\mathcal{G}, \mathcal{G}'} U)^{n+1} P_{\frac{\bigtriangleup}{2}}(\frac{\bigtriangleup}{2})
\]
for each graph \((n+1)\)-simplex \( \mathcal{H} \in \mathcal{D}^{n+1}(\frac{\bigtriangleup}{2}) \). We will use these natural maps in Prop. 9.1.19 below when we describe the simplicial structure maps of the bar resolution.

Suppose \((L, \varepsilon, \delta)\) is a comonad on a general category with counit \( \varepsilon : L \longrightarrow \text{Id} \) and comultiplication \( \delta : L \longrightarrow L^2 \). There is a simplicial endofunctor \( L_{**1} \) with face and degeneracy maps
\[
\begin{align*}
&d_i = L^i \varepsilon L^{n-i} : L^{n+1} \longrightarrow L^n, \quad i = 0, \ldots, n, \\
ds_i = L^i \delta L^{n-i-1} : L^n \longrightarrow L^{n+1}, \quad i = 0, \ldots, n-1.
\end{align*}
\]

See, e.g., [Mac98] (page 181) or [Wei97] (8.6.4 page 280). To make precise these simplicial structure maps when the comonad comes from the free-forgetful adjunction \( F^{\mathcal{G}, \mathcal{G}'} U \), we need the following definition.

**Definition 9.1.17.** Suppose \( \mathcal{G} \leq \mathcal{G}' \) is an inclusion of \( \mathcal{C} \)-colored pasting schemes, and \( H = H^{[1,n+1]} \) is a graph \((n+1)\)-simplex with profiles \( \frac{\bigtriangleup}{2} \) for some \( n \geq 0 \). Suppose \( 0 \leq i \leq n \).

1. For \( n \geq 1 \), define the graph \( n \)-simplex
\[
d_i H = \begin{cases} 
(H^1, \ldots, H^{n-i+1}(H^{n-i}), \ldots, H^{n+1}) & \text{if } 0 \leq i \leq n-1, \\
(H^2, \ldots, H^{n+1}) & \text{if } i = n.
\end{cases}
\]
Here $H^{n-i+1}(H^{n-i})$ is the family of graphs obtained by performing all possible graph substitutions in the two indicated layers.

(2) Define the graph $(n+2)$-simplex

$$s_i H = \left( H^1, \ldots, H^{n-i}, C, H^{n-i+1}, \ldots, H^{n+1} \right),$$

where $C$ is the family of corollas indexed by the set of vertices in $H^{n-i+1}$ such that each corolla $C_v$ has the same profiles as the indexing vertex $v \in H^{n-i+1}$.

**Example 9.1.18.** For a graph 3-simplex $H = (H^1, H^2, H^3)$, the above maps are given as follows.

$$d_0 H = (H^1, H^3(H^2)) \quad s_0 H = (H^1, H^2, \{C_v\}_{v \in H^2}, H^3)$$

$$d_1 H = (H^2(H^1), H^3) \quad s_1 H = (H^1, \{C_v\}_{v \in H^2}, H^2, H^3)$$

$$d_2 H = (H^2, H^3) \quad s_2 H = (\{C_v\}_{v \in H^2}, H^1, H^2, H^3).$$

Each $d_i H$ is a graph 2-simplex, and each $s_i H$ is a graph 4-simplex. Moreover, these maps satisfy the simplicial identities. This is no surprise in view of the next result.

The following observation makes explicit the face and degeneracy maps in the bar resolution.

**Proposition 9.1.19.** Suppose $G \leq G'$ is an inclusion of $G$-colored pasting schemes and $P$ is a $G'$-prop in $M$. Consider a pair of $\mathcal{C}$-profles $(\mathcal{G})$ and the simplicial $G'$-prop $(F^G, G') \ast P$.

(1) For $n \geq 1$ and $0 \leq i \leq n$, the face maps are determined by the commutative diagram

$$
\begin{array}{ccc}
P[d_i H] & \xleftarrow{\otimes_{n-i}^p} & P[H] \\
\downarrow & & \downarrow \\
(F^G, G')^n(\mathcal{G}) & \xleftarrow{d_i} & (F^G, G')^{n+1}(\mathcal{G}) \\
\downarrow & & \downarrow \\
P[d_i H] & \xleftarrow{\gamma} & P[H]
\end{array}
$$

for $H \in D^{n+1}(\mathcal{G})$ in which all vertical maps are natural maps. The top horizontal map is induced by the $G'$-prop structure maps $\gamma^p_H$ for $H \in H^1$.

(2) For $0 \leq i \leq n$ the degeneracy maps are determined by the commutative diagram

$$
\begin{array}{ccc}
(F^G, G')^{n+1}(\mathcal{G}) & \xrightarrow{s_i} & (F^G, G')^{n+2}(\mathcal{G}) \\
\downarrow & & \downarrow \\
P[H] & \xrightarrow{\gamma} & P[s_i H]
\end{array}
$$

in which both vertical maps are natural maps.

**Proof.** Simply apply (9.1.16) to the comonad $L = F^G, G' U$ with:
9.2. Free Generalized Prop of the Boardman-Vogt Construction

For a $G$-prop $P$ in $M$, the $W$-construction $W(G, J, P)$ involves a choice of a commutative segment $J$ in $M$. The main result of this section is Theorem 9.2.12. It shows that, for a shrinkable pasting scheme (Def. 9.2.5), when the functor $F^G U$ from the free-forgetful adjunction (9.1.2) is applied to the $W$-construction $W(G, J, P)$, the result is also a $W$-construction $W(G, J^*, P)$ with a different commutative segment $J^*$ (Def. 9.2.1). This is an important step in identifying the bar resolution of a $G$-prop with an instance of the $W$-construction. Since the pasting scheme $G$ is fixed, we will drop it from the notation of the $W$-construction, so

$$W(J, P) = W(G, J, P).$$

The main result of this section involves the following construction $J^*$, which is from [BM06] (Def. 6.2).

**Definition 9.2.1.** Suppose $(J, \mu, 0, 1, \epsilon)$ is a commutative segment in $M$. Define a commutative segment

$$(J^* = J \sqcup 1, \mu^*, 0, \text{Id}, (\epsilon, \text{Id}))$$

with structure maps

$$1 \sqcup 1 \xrightarrow{(0, \text{Id})} J \sqcup 1 \xrightarrow{((\epsilon, \text{Id})} 1$$

and multiplication

$$(J \sqcup 1) \otimes (J \sqcup 1) \cong J^{\otimes 2} \sqcup (1 \otimes J) \sqcup (J \otimes 1) \sqcup 1^{\otimes 2}$$

$$\mu^* = \begin{cases} (\mu, \epsilon, c, \bar{z}) \\ J \sqcup 1. \end{cases}$$

In other words, $J^* = J \sqcup 1$ is $J$ with a newly adjoined absorbing element.
Remark 9.2.2. The inclusion map $J \rightarrow J_*$ preserves all the structure except for the absorbing element. On the other hand, the map

$$J_* = J \cup I \xrightarrow{(\text{Id}, \text{Id})} J$$

is a map of commutative segments.

Consider the commutative segment $I$ with:

- the identity map as the structure maps $0$, $1$, and $\epsilon$;
- the natural isomorphism $I \otimes I \cong I$ as the multiplication.

Example 9.2.3. Then $I_* = I_0 \cup I_1$ is the commutative segment with structure maps

$$I_0 \cup I_1 \xrightarrow{\text{Id}} I_0 \cup I_1 \xrightarrow{(\text{Id,Id})} I$$

and multiplication

$$(I_0 \cup I_1) \otimes (I_0 \cup I_1) \cong (I_0 \otimes I_0) \cup (I_1 \otimes I_0) \cup (I_0 \otimes I_1) \cup (I_1 \otimes I_1)$$

$$(0,1,1,1)$$

$$I_0 \otimes I_1.$$ In other words, the multiplication is determined by the maximum operation on $\{0,1\}$ in the subscripts.

Example 9.2.4. Iterating the construction one more time, the commutative segment

$$(I_*)_* = (I_0 \cup I_1)_* = I_0 \cup I_1 \cup I_2$$

has structure maps

$$I_0 \cup I_2 \xrightarrow{\text{inclusion}} I_0 \cup I_1 \cup I_2 \xrightarrow{(\text{Id,Id,Id})} I$$

and multiplication

$$(I_0 \cup I_1 \cup I_2)^{\otimes 2} \rightarrow I_0 \cup I_1 \cup I_2$$

determined by the maximum operation on $\{0,1,2\}$ in the subscripts. Note that the absorbing element is $I_2$, i.e., the copy of $I$ with the largest subscript. These examples and their higher analogues will play an important role in Section 9.3 and later.

Recall that $|G|$ denotes the set of ordinary internal edges in a graph $G$. The next definition of a shrinkable pasting scheme is based on a similar one in [HRY17].

Definition 9.2.5. A pasting scheme $G$ is shrinkable if:

1. It is connected.
2. For each $G \in \mathcal{G}$ and each partition $|G| = A \cup B$

there exists a graph substitution decomposition

$$G = K(\mathcal{H}_v)_{v \in K}$$

in $\mathcal{G}$ such that

$$|K| = A \quad \text{and} \quad \bigsqcup_{v \in K} |\mathcal{H}_v| = B.$$

Lemma 9.2.7. Suppose $\mathcal{G}$ is a shrinkable $\mathcal{C}$-colored pasting scheme, and $G \in \mathcal{G}(\mathcal{G})$ with a partition $|G| = A \cup B$. Then the graph substitution decomposition $G = K(H_v)_{v \in K}$ in \[9.2.6\] is unique up to vertex profile permutations in $K$ and corresponding graph profile permutations in the $H_v$’s.

Proof. Connectivity implies that:

(1) $K$ must be obtained from $G$ by shrinking away the ordinary internal edges in $B \subseteq |G|$.

(2) The $H_v$’s must consist of those graphs that are shrunk away in the previous step.

The graph profiles of $K$ must coincide with those of $G$, and the collective vertex profiles of the $H_v$’s must coincide with those of $G$. The only choices involved are the vertex profiles of $K$ and corresponding the graph profiles of the $H_v$’s. \hfill \Box

Proposition 9.2.8. The pasting schemes of

1. unital linear graphs (for small categories),
2. unital trees (for operads),
3. simply-connected graphs (for dioperads),
4. wheeled trees (for wheeled operads), and
5. connected wheeled graphs (for wheeled properads)

are shrinkable.

Proof. These pasting schemes are connected by definition. To produce the required graph substitution decomposition \[9.2.6\], apply the procedure described in the proof of Lemma 9.2.7. \hfill \Box

Example 9.2.9. Consider the following connected wheeled graph $G$

with three internal edges $|G| = \{a, b, c\}$.

1. Consider the partition $|G| = \{b, c\} \cup \{a\}$.

Then $G$ decomposes as $G = K_1(H_1)$ with $K_1$ and $H_1$ depicted as follows.

So $|K_1| = \{b, c\}$ and $|H_1| = \{a\}$. With these properties, the only changes we can make to the decomposition $G = K_1(H_1)$ is to permute the vertex profiles at $w$ and correspondingly permute the graph profiles of $H_1$. 

\hfill \Box
(2) Consider the partition
\[ |G| = \{a\} \cup \{b, c\}. \]

Then \( G \) decomposes as \( G = K_2(H_2) \) with \( K_2 \) and \( H_2 \) depicted as follows.

\[ K_2 \]
\[ H_2 \]

So \( |K_2| = \{a\} \) and \( |H_2| = \{b, c\} \). With these properties, the only changes we can make to the decomposition \( G = K_2(H_2) \) is to permute the vertex profiles at \( w \) and correspondingly permute the graph profiles of \( H_2 \).

**Example 9.2.10.** On the other hand, the pasting scheme of connected wheel-free graphs (for properads) is *not* shrinkable, since shrinking away either internal edge in the graph

would yield a single vertex with a loop. Similarly, the pasting schemes of all wheeled graphs (for wheeled props) and of wheel-free graphs (for props) are also not shrinkable because they are not connected.

**Motivation 9.2.11.** For a \( \mathcal{G} \)-prop \( P \), by definition the \( W \)-construction \( W(J, P) \) is entrywise a space of \((J, P)\)-decorated graphs with relations parametrized by the substitution categories. For a \( \mathcal{G}_0 \)-prop \( X \), the free \( \mathcal{G} \)-prop \( F^\mathcal{G}X \) is entrywise a space of \( X \)-decorated graphs with relations parametrized by the extension category. So \( F^\mathcal{G}UW(J, P) \) is entrywise a space of graphs (“outside graphs”) whose vertices are decorated by \((J, P)\)-decorated graphs. Thinking of the ordinary internal edges of the outside graphs as having length 1, \( F^\mathcal{G}UW(J, P) \) should consist of \((J \cup 1, P)\)-decorated graphs. In other words, it should be the \( W \)-construction of \( P \) using the commutative segment \( J_* = J \cup 1 \). The following result verifies this guess precisely.

**Theorem 9.2.12.** Suppose \( \mathcal{G} \) is a shrinkable \( \mathcal{C} \)-colored pasting scheme, and \( P \) is a \( \mathcal{G} \)-prop in \( \mathcal{M} \), which is equipped with a commutative segment \( J \). Then there is a unique isomorphism
\[
F^\mathcal{G}UW(J, P) \cong W(J_*, P)
\]
of \( \mathcal{G} \)-props in \( \mathcal{M} \), where \( J_* = J \cup 1 \) (Def. [9.2.1]).

**Proof.** Since graphs in \( \mathcal{G}_0 \) do not have ordinary internal edges, the \( \mathcal{G}_0 \)-prop structure of \( W(?, P) \) does not involve the absorbing element of the commutative segment being used. So the inclusion \( J \to J \cup 1 \), which does not preserve the absorbing element, still defines a map
\[
W(J, P) \to W(J_*, P)
\]
of underlying \( \mathcal{G}_0 \)-props.

Since \( F^\mathcal{G} \) is a left adjoint, when applied to the underlying \( \mathcal{G}_0 \)-prop of \( W(J, P) \), it is uniquely characterized by the universal property:
For each $G$-prop $Y$, each map
\[ \varphi : W(J, P) \longrightarrow Y \]
of underlying $G_0$-props extends uniquely to a $G$-prop map
\[ \overline{\varphi} : F^G UW(J, P) \longrightarrow Y. \]

We will show that $W(J_*, P)$ also has this universal property with respect to the $G_0$-prop map $W(J, P) \longrightarrow W(J_*, P)$ from the first paragraph.

For each graph $G \in G(\mathcal{E})$, we have that
\[ (9.2.13) \quad J_*[G] = \bigotimes_{e \in |G|} (J \cup \mathbb{1}) \cong \bigoplus_{f \in \{0,1\}^{|G|}} \left( \bigotimes_{f=0} J \otimes \mathbb{1} \right). \]

Here $\{0,1\}^{|G|}$ is the set of functions $|G| \longrightarrow \{0,1\}$. For a function $f : |G| \longrightarrow \{0,1\}$ and $i \in \{0,1\}$, the notation $\bigotimes_{f=i}$ means the tensor product is indexed by those ordinary internal edges $e$ in $G$ such that $f(e) = i$. Tensoring with $P[G]$ yields
\[ (9.2.14) \quad J_*[G] \otimes P[G] \cong \bigoplus_{f \in \{0,1\}^{|G|}} \left( \bigotimes_{f=0} J \right) \otimes P[G]. \]

Suppose given a $G$-prop $Y$ and a map
\[ \varphi : W(J, P) \longrightarrow Y \]
of underlying $G_0$-props. Since the desired extension $\overline{\varphi}$ must restrict to $\varphi$ and must be compatible with the $G$-prop structure maps, there is only one possible candidate for $\overline{\varphi}$. Namely, for each graph $K \in G(\mathcal{E})$ and each function $f : |K| \longrightarrow \{0,1\}$, we must defined $\overline{\varphi}$ entrywise by declaring that the diagram
\[ (9.2.15) \quad \begin{array}{ccc}
\left( \bigotimes_{f=0} J \right) \otimes P[K] & \xrightarrow{z} & \bigotimes_{u \in D} \left( J \otimes P \right)[E_u] \otimes \omega_{E_u} \\
\downarrow f \text{ summand} & & \downarrow \otimes \varphi \\
(J_* \otimes P)[K] & \xrightarrow{\omega_K} & \bigotimes_{u \in D} Y(u) = Y[D] \\
\downarrow \omega_K & & \downarrow \gamma_D \rightarrow \theta(Y) \\
W(J_*, P)(\mathcal{E}) & \xrightarrow{\overline{\varphi}} & Y(\mathcal{E})
\end{array} \]
in $M$ be commutative. Here
\[ K = D(E_u)_{u \in D} \]
is the decomposition \[9.2.6\] such that
\[ (9.2.16) \quad |D| = \{ e \in |K| : f(e) = 1 \} \quad \text{and} \quad \bigcup_{u \in D} |E_u| = \{ e \in |K| : f(e) = 0 \}. \]

In Lemma \[9.2.18\] below we will show that $\overline{\varphi}$ is entrywise a well-defined map. In Lemma \[9.2.21\] we will show that $\overline{\varphi}$ is a map of $G$-props. Since we already noted that $\overline{\varphi}$ is unique if it is defined, it remains to prove these two Lemmas. \qed
Example 9.2.17. Consider the connected wheeled graph $G$ with $|G| = \{a, b, c\}$ in Example 9.2.9.

1. The function $f : |G| \to \{0, 1\}$ with
   \[
   f(a) = 0 \quad \text{and} \quad f(b) = f(c) = 1
   \]
corresponds to the decomposition $G = K_1(H_1)$, so
   \[
   \bigotimes_{f=0} J = J.
   \]

2. The function $f' : |G| \to \{0, 1\}$ with
   \[
   f'(a) = 1 \quad \text{and} \quad f'(b) = f'(c) = 0
   \]
corresponds to the decomposition $G = K_2(H_2)$, so
   \[
   \bigotimes_{f'=0} J = J \otimes J.
   \]

Lemma 9.2.18. The map
   \[
   \varphi : W(J_*P)(\frac{\partial}{\partial}) \to Y(\frac{\partial}{\partial})
   \]
in (9.2.15) is well-defined.

Proof. Suppose
   \[
   G = K(H_{e}) \xrightarrow{(H_{v})_{v \in K}} K
   \]
is a map in $\mathcal{G}(\frac{\partial}{\partial})$. We must show that the diagram
   \[
   \begin{array}{ccc}
   J_*[K] \otimes P[G] & \xrightarrow{\otimes_{v \in K} \mu_v} & J_*[K] \otimes P[K] \\
   J_*[G] \otimes P[G] & \xrightarrow{\varphi_{v \in G}} & Y(\frac{\partial}{\partial})
   \end{array}
   \]
is commutative. Since $\varphi$ is a map of $\mathcal{G}_0$-props and $\mathcal{G}$ is connected, we may assume that each $H_{e}$ is an ordinary graph. Using the decomposition for $J_*[K]$ in (9.2.13) with $K$ in place of $G$, it is enough to show that the previous square is commutative when restricted to a typical $f$ summand of the upper left corner for a function $f : |K| \to \{0, 1\}$. Suppose $K = D(E_{u})$ is the decomposition (9.2.16) corresponding to $f$.

Extend $f$ to a function $g : |G| \to \{0, 1\}$ by declaring
   \[
   g(e) = \begin{cases} 
   f(e) & \text{if } e \in |K|; \\
   0 & \text{if } e \in \bigsqcup_{v \in K} |H_{v}|.
   \end{cases}
   \]

For each vertex $u \in D$, there is a map
   \[
   E_{u}(H_{v})_{v \in E_{u}} \xrightarrow{(H_{v})_{v \in E_{u}}} E_{u}
   \]
in $\mathcal{G}(u)$. Also, we have that
   \[
   \bigotimes_{f=0} J = \bigotimes_{u \in D} J[E_{u}] \quad \text{and} \quad \bigotimes_{g=0} J = \bigotimes_{u \in D} J[E_{u}(H_{v})].
   \]
The restriction of the diagram (9.2.19) to the $f$ summand is the outer diagram in:

\[
\begin{array}{c}
\bigotimes_{v \in K} J \otimes P[G] \xrightarrow{\varphi \otimes P[K]} \bigotimes_{v \in D} J \otimes P[E_u] \xrightarrow{\varphi \otimes P[K]} \bigotimes_{v \in D} J \otimes P[E_u(H_v)] \xrightarrow{\varphi \otimes P[K]} \bigotimes_{v \in D} J \otimes P[P]\end{array}
\]

\[
\begin{array}{c}
\bigotimes_{v \in D} J \otimes P[G] \xrightarrow{\varphi \otimes P[K]} \bigotimes_{v \in D} J \otimes P[E_u] \xrightarrow{\varphi \otimes P[K]} \bigotimes_{v \in D} J \otimes P[E_u(H_v)] \xrightarrow{\varphi \otimes P[K]} \bigotimes_{v \in D} J \otimes P[P]\end{array}
\]

In the upper left vertical map 0 is the map $0 \xrightarrow{1} J$. The sub-diagram 1 is the tensor product over vertices $u \in D$ of diagrams that are commutative by the coend definition of $W(J, P)(u)$ and the maps in (9.2.20). The diagrams 2 and 3 are simply the definitions of $\varphi$ when restricted to the $g$ and $f$ summands. The sub-diagrams 4 and 5 are commutative by definition. \hfill \Box

**Lemma 9.2.21.** The maps $\varphi$ in (9.2.15) assemble to a map $W(J_*, P) \longrightarrow Y$ of $G$-props.

**Proof.** Suppose $G \in \mathcal{G}(\varphi)$ for some pair $\varphi(\varphi)$ of $C$-profiles. We must show that the diagram

\[
\begin{array}{c}
W(J_*, P)[G] = \bigotimes_{v \in G} W(J_*, P)(v) \xrightarrow{\varphi \otimes \gamma_{W(J_*, P)}} \bigotimes_{v \in G} Y(v) = Y[G] \\
\end{array}
\]

is commutative.

For each vertex $v \in G$, pick $H_v \in \mathcal{G}(v)$ and $f_v : |H_v| \longrightarrow \{0, 1\}$. Since the upper left corner of the previous diagram is a colimit, using the decomposition (9.2.14) for the $H_v$'s, it is enough to show the commutativity of the previous diagram when restricted to the object

\[
\bigotimes_{v \in G} (\bigotimes_{f_v=0} J \otimes P[H_v]).
\]

Moreover, since $\varphi$ is a map of $G_0$-props and $G$ is connected, we may assume that $G$ and all the $H_v$'s are ordinary graphs. Since $G$ is shrinkable, for each vertex $v \in G,$
there is a decomposition (9.2.6)

\[ H_v = D_v(E_w) \]

in \( G \) with

\[
|D_v| = \{ e \in |H_v| : f_v(e) = 1 \} \quad \text{and} \quad \prod_{w \in D_v} |E_w| = \{ e \in |H_v| : f_v(e) = 0 \}
\]

Extend \( \{f_v\}_{v \in G} \) to \( g : |G(H_v)| \to \{0, 1\} \) by setting

\[
g(e) = \begin{cases} f_v(e) & \text{if } e \in |H_v|, \\ 1 & \text{if } e \in |G|. \end{cases}
\]

The diagram (9.2.22) restricted to the object (9.2.23) is the outer diagram in:

The sub-diagram 1 is commutative by the definitions of \( \pi \) (3.5.6) and of \( g \). The sub-diagram 2 is commutative by the definition of the \( G \)-prop structure map \( \gamma_G \) in \( W(J_*, P) \) (3.5.5). The sub-diagram 3 is the definition of \( \varphi \) restricted to the \( g \) summand. The sub-diagram 4 is commutative by the associativity of the \( G \)-prop structure map of \( Y \).

The proof of Theorem 9.2.12 is now complete.

**Remark 9.2.24.** Observe that the isomorphism

\[ F^G \text{UW}(G, J, P) \cong W(J_*, P) \]
9.3. Bar Resolution is Levelwise a Boardman-Vogt Construction

Using Theorem 9.2.12 here we observe that each level of the bar resolution is an instance of the $W$-construction with respect to an appropriate choice of a commutative segment. See Theorem 9.3.6. It is important to note that different commutative segments are used for different levels of the bar resolution. Eventually this sequence of $W$-constructions will be assembled to form a single $W$-construction in simplicial objects in $M$. We will also record some other interesting consequences of Theorem 9.2.12.

Consider the commutative segment $1$ with the identity map as 0, and the counit and with the natural isomorphism $1 \otimes 1 \cong 1$ as the multiplication.

**Definition 9.3.1.** For $n \geq 0$ define $1_{[0,n+1]}$ as the commutative segment obtained from $1$ by applying $?_{*}$ in Def. 9.2.1 $n+1$ times.

**Example 9.3.2.** The commutative segment $1_{[0,1]}$ is $1_{*} = 1_{0} \cup 1_{1}$ in Example 9.2.3. The commutative segment $1_{[0,2]}$ is $1_{0} \cup 1_{1} \cup 1_{2}$ in Example 9.2.4.

**Lemma 9.3.3.** As an object in $M$, we have that

$$
1_{[0,n+1]} = \bigsqcup_{i=0}^{n+1} 1_{i},
$$

with each $1_{i}$ a copy of $1$. Its commutative segment structure maps are

$$
1 \sqcup 1 \xrightarrow{(0,n+1)} \bigsqcup_{i=0}^{n+1} 1_{i} \xrightarrow{\text{fold}} 1
$$

with multiplication determined by the maximum operation on indices:

$$
\left( \bigsqcup_{i=0}^{n+1} 1_{i} \right)^{\otimes 2} \equiv \bigsqcup_{i,j=0}^{n+1} 1_{i} \otimes 1_{j} \xrightarrow{\mu} \bigsqcup_{i=0}^{n+1} 1_{i}
$$

$$
\text{inclusion} \quad 1_{i} \otimes 1_{j} \xrightarrow{\cong} 1_{\max(i,j)}.
$$

**Proof.** This is a simple induction on $n \geq 0$, where both the initial case and the induction step follow directly from Def. 9.2.1. \qed
Remark 9.3.4. In the commutative segment $\mathbb{1}_{[0,n+1]}$, the neutral element is $\mathbb{1}_0$, and the absorbing element is $\mathbb{1}_{n+1}$.

The next preliminary result says that a $\mathcal{G}$-prop can be regarded as a $W$-construction.

**Proposition 9.3.5.** For each $\mathcal{G}$-prop $P$ in $\mathcal{M}$, the augmentation (Prop. 4.1.2)

$$W(\mathbb{1}, P) \xrightarrow{\eta^p} P$$

is an isomorphism of $\mathcal{G}$-props.

**Proof.** For each pair $(\bar{d}, \bar{c})$ of $\mathcal{C}$-profiles, we just need to see that the augmentation

$$\eta : W(\mathbb{1}, P)(\bar{d}) \longrightarrow P(\bar{d})$$

is an isomorphism in $\mathcal{M}$. Indeed, the maps

$$\mathbb{1}[G] \otimes P[G] \cong P[G] \xrightarrow{\gamma^p_G} P(\bar{d}) \quad \text{for} \quad G \in G(\bar{d})$$

give $P(\bar{d})$ the same universal property as the coend

$$W(\mathbb{1}, P)(\bar{d}) = \int^{G \in G(\bar{d})} \mathbb{1}[G] \otimes P[G]$$

because the contravariant variable $\mathbb{1}$ is mute. □

**Theorem 9.3.6.** Suppose $\mathcal{G}$ is a shrinkable $\mathcal{C}$-colored pasting scheme, and $P$ is a $\mathcal{G}$-prop in $\mathcal{M}$. For each $n \geq 0$, Theorem 9.2.12 yields a canonical isomorphism

$$(F^G U)^{n+1} P \cong W(\mathbb{1}_{[0,n+1]}, P)$$

of $\mathcal{G}$-props in $\mathcal{M}$.

**Proof.** The proof proceeds by induction on $n$. For $n = 0$, Prop. 9.3.5 and Theorem 9.2.12 with $J = \mathbb{1}$ yield the canonical isomorphisms

$$F^G U P \cong F^G UW(\mathbb{1}, P) \cong W(\mathbb{1}_*, P),$$

and

$$\mathbb{1}_* = \mathbb{1}_{[0,1]}$$

by definition.

Suppose $n \geq 1$ and that the $n$ case is true. The induction hypothesis and Theorem 9.2.12 with $J = \mathbb{1}_{[0,n]}$ yield the canonical isomorphisms

$$(F^G U)^{n+1} P = (F^G U)(F^G U)^nP$$

$$\cong F^G UW(\mathbb{1}_{[0,n]}, P)$$

and

$$(\mathbb{1}_{[0,n]})_* = \mathbb{1}_{[0,n+1]}$$

by definition. □
Remark 9.3.7. In order to identify the bar resolution of a \( G \)-prop with an instance of the \( W \)-construction, it is important to make the isomorphism in the previous result explicit. Using (9.2.25) repeatedly starting with \( J = \mathbb{1} \), the isomorphism

\[
(F^G U)^{n+1} P \cong W(\mathbb{1}_{[0,n+1]}, P)
\]

is entrywise uniquely determined by the commutative diagrams

\[
\begin{array}{ccc}
P[H] & \xrightarrow{h} & (\mathbb{1}_{[0,n+1]} \otimes P)[\text{sub}(H)] \\
\downarrow \text{natural} & & \downarrow \omega_{\text{sub}(H)} \\
(F^G U)^{n+1} P(\mathbb{1}) & \xrightarrow{\varphi} & W(\mathbb{1}_{[0,n+1]}, P)(\mathbb{1})
\end{array}
\]

for graph \((n + 1)\text{-simplices } H = H^{[1,n+1]} \in E^{n+1}(\mathbb{1})\). On the left side we used the colimit in Prop. 9.1.14. The top horizontal map \( h \) is

\[
h = \bigotimes_{i=1}^{n+1} \bigotimes_{e \in [H]} (1_i \xrightarrow{\text{inclusion}} \mathbb{1}_{[0,n+1]}).
\]

In other words, for each \( 1 \leq i \leq n \), the ordinary internal edges in the layer \( H^i \) are assigned the label \( i \). In particular, no ordinary internal edges are assigned the label 0 by the map \( h \).

The following observation implies, in particular, that the counit map of a \( G \)-prop is induced by a change of commutative segments.

Corollary 9.3.10. Suppose \( G \) is a shrinkable \( C \)-colored pasting scheme, and \( P \) is a \( G \)-prop in \( \mathcal{M} \) equipped with a commutative segment \( J \). Then the \( G \)-prop maps

\[
W(\mathbb{1}_{[0,1]}, P) \longrightarrow W(J, P) \longrightarrow W(\mathbb{1}, P)
\]

induced by the maps

\[
\mathbb{1}_0 \sqcup \mathbb{1}_1 \xrightarrow{(0,1)} J \xrightarrow{\sigma} \mathbb{1}
\]

of commutative segments are canonically isomorphic to the maps

\[
F^G U P \xrightarrow{\delta} W(J, P) \xrightarrow{\eta^P} P
\]

in Theorem 4.2.14 and Prop. 4.1.2.

Proof. Apply the \( n = 0 \) case of Theorem 9.3.6 and Prop. 9.3.5.

Corollary 9.3.11. Suppose \( G \) is a shrinkable \( C \)-colored pasting scheme, and \( P \) is a \( G \)-prop in \( \mathcal{M} \) equipped with a commutative segment \( J \). Then the change of segment map

\[
W(J, P) \longrightarrow W(\mathbb{1}, P)
\]

induced by the map

\[
J_\ast = J \sqcup \mathbb{1} \xrightarrow{(\text{Id}, 1)} J
\]

of commutative segments is canonically isomorphic to the counit

\[
F^G U W(J, P) \longrightarrow W(J, P)
\]

of the adjunction applied to \( W(J, P) \).
Proof. The counit of a $G$-prop, such as $W(J, P)$, is induced by the $G$-prop structure maps as in (9.1.4). The assertion now follows from the explicit description of the isomorphism

$$F^G U W(J, P) \cong W(J, P)$$

in (9.2.25) and the definition of the $G$-prop structure in $W(J, P)$ (3.5.5).

9.4. Boardman-Vogt Resolution in Simplicial Objects

Our purposes here are to:

1. Observe that the sequence of $W$-constructions $W(\mathbb{I}_{[0, n+1], P})$ for $n \geq 0$ in Theorem 9.3.6 fit together to form a single $W$-construction in simplicial objects in $M$;

2. Describe the corresponding simplicial structure maps explicitly.

In the next section, the resulting $W$-construction is then identified with the bar resolution $(F^G U)^{+1} P$. Furthermore, a suitable realization of this identification at the simplicial level shows that a realization of the bar resolution is canonically isomorphic to a $W$-construction of $P$ in $M$. To talk about $W$-constructions in simplicial objects in $M$, we first need to specify a commutative segment there. We will transport the standard commutative segment in simplicial sets to simplicial objects in $M$.

Recall the category $\Delta$ whose objects are totally ordered sets

$$[n] = \{0 < 1 < \cdots < n\}$$

for $n \geq 0$ and whose morphisms are weakly order-preserving maps. A simplicial object in $M$ is a functor $\Delta^{\text{op}} \to M$. The category of simplicial objects in $M$ with natural transformations as maps is denoted by $sM$. The representable simplicial set

$$\Delta^1 = \Delta(\{\}, [1])$$

is a commutative segment in the category $\mathbf{SSet}$ of simplicial sets with structure maps

$$\Delta^0 \sqcup \Delta^0 \xrightarrow{(0,1)} \Delta^1 \xrightarrow{\epsilon} \Delta^0$$

and multiplication induced by the maximum operation on $\{0, 1\}$.

Definition 9.4.1. Denote by $\Delta^1_M$ the commutative segment in $sM$ obtained by applying the strong symmetric monoidal functor

$$\text{Set} \to M, \quad T \mapsto T_M = \bigsqcup_{T} \mathbb{I}$$

to $\Delta^1$. We may also think of $\Delta^1_M$ as a simplicial object in commutative segments in $M$.

Lemma 9.4.2. The object of $n$-simplices in $\Delta^1_M$ is the commutative segment $\mathbb{I}_{[0, n+1]}$ in Def. 9.3.1.

Proof. As an object in $M$, we have that

$$(\Delta^1_M)^n = \bigsqcup_{\Delta([n],[1])} \mathbb{I} = \bigsqcup_{i=0}^{n+1} \mathbb{I}_{i}.$$
the last \( i \) objects (namely, \( n - i + 1, \ldots, n \)) to \( 1 \in [1] \) and

- the first \( n - i + 1 \) objects (namely, \( 0, \ldots, n - i \)) to \( 0 \in [1] \).

As the commutative segment structure of \( \Delta^1_M \) is induced by the maximum operation on \( \{0, 1\} \), it corresponds to that of \( 1_{[0, n+1]} \) as described in Lemma 9.3.3. □

**Example 9.4.3.** The multiplication on \((\Delta^1_M)_n\) sends \( 1_i \otimes 1_j \) to the copy of \( 1 \) corresponding to the function \([n] \to [1]\) that sends the last \( \max\{i, j\} \) objects to \( 1 \in [1] \), i.e., \( 1_{\max\{i, j\}} \). This is the same as the multiplication in \( 1_{[0, n+1]} \). Under this multiplication:

- \( 1_0 \) (corresponding to the function that sends all of \([n]\) to \( 0 \in [1] \)) is neutral.
- \( 1_{n+1} \) (corresponding to the function that sends all of \([n]\) to \( 1 \in [1] \)) is absorbing.

We also need to make explicit the simplicial structure maps of \( \Delta^1_M \).

**Definition 9.4.4.** For \( 0 \leq i \leq n \) define the functions:

1. \( \partial_i : [n+1] \to [n] \) by
   \[
   \partial_i(j) = \begin{cases} 
   j & \text{if } 0 \leq j \leq n - i, \\
   j - 1 & \text{if } n - i + 1 \leq j \leq n + 1.
   \end{cases}
   \]

2. \( \sigma_i : [n+1] \to [n+2] \) by
   \[
   \sigma_i(j) = \begin{cases} 
   j & \text{if } 0 \leq j \leq n - i, \\
   j + 1 & \text{if } n - i + 1 \leq j \leq n + 1.
   \end{cases}
   \]

**Lemma 9.4.5.** Under the identification of Lemma 9.4.2, for \( 0 \leq i \leq n \):

1. There are the face maps
   \[
   (\Delta^1_M)_n = \prod_{j=0}^{n+1} 1_j \quad \xrightarrow{d_i} \quad 1_j \quad \xleftarrow{\partial_i(j)} \quad (\Delta^1_M)_{n-1} = \prod_{j=0}^{n} 1_j.
   \]

2. There are the degeneracy maps
   \[
   (\Delta^1_M)_{n+1} = \prod_{j=0}^{n+2} 1_j \quad \xrightarrow{s_i} \quad 1_{\sigma_i(j)} \quad \xleftarrow{s_i} \quad (\Delta^1_M)_n = \prod_{j=0}^{n+1} 1_j.
   \]

**Proof.** The simplicial structure on \( \Delta^1_M \) comes from that on \( \Delta^1 \). The face map
   \[
   d_i : \Delta^1_n \to \Delta^1_{n-1}
   \]

is given by pre-composition with the coface map
   \[
   d^i : [n-1] \to [n]
   \]
in $\Delta$ that skips $i$. The degeneracy map
\[ s_i : \Delta^1_n \to \Delta^1_{n+1} \]
is given by pre-composition with the codegeneracy map
\[ s^i : [n+1] \to [n] \]
that doubles up on $i$. An inspection shows that these simplicial structure maps are the stated ones above.

\[ \square \]

**Example 9.4.6.** The coface map
\[ d^n : [n-1] \to [n] \]
skips $n$, so $d^n(k) = k$ for each $0 \leq k \leq n-1$.

1. In $(\Delta^1_M)_n$ the copy $1_0$ corresponds to the function $[n] \to [1]$ that sends all of $[n]$ to 0 $\in [1]$. Pre-composing this function with the coface map $d^n$, we see that $d_n 1_0$ corresponds to the function $[n-1] \to [1]$ that sends all of $[n-1]$ to 0 $\in [1]$, i.e.,
\[ d_n 1_0 = 1_0. \]

2. For $1 \leq j \leq n+1$, the copy $1_j$ in $(\Delta^1_M)_n$ corresponds to the function $[n] \to [1]$ that sends the last $j$ objects in $[n]$ (i.e., $n-j+1,\ldots,n$) to 1 $\in [1]$. Pre-composing this function with the coface map $d^n$, we see that $d_n 1_j$ corresponds to the function $[n-1] \to [1]$ that sends the last $j-1$ objects in $[n-1]$ (i.e., $n-j+1,\ldots,n-1$) to 1 $\in [1]$, i.e.,
\[ d_n 1_j = 1_{j-1}. \]

Taken together we infer that
\[ d_n 1_j = 1_{d_n(j)} \]
for $0 \leq j \leq n+1$.

The constant simplicial object functor
\[ M \to sM \]
is strong symmetric monoidal when $sM$ is equipped with the levelwise monoidal structure:
\[ (X \otimes Y)_n = X_n \otimes Y_n. \]

**Definition 9.4.7.** Suppose $P$ is a $G$-prop in $M$, which is also regarded as a $G$-prop in $sM$ via the constant simplicial object functor $M \to sM$. Using the commutative segment $\Delta^1_M$ in $sM$, we define the $W$-construction
\[ W(\Delta^1_M, P) \in \text{Prop}^G(sM), \]
which may also be regarded as a simplicial object in $\text{Prop}^G(M)$. By Lemma 9.4.2 its object of $n$-simplices is the $G$-prop
\[ W(\Delta^1_M, P)_n = W((\Delta^1_M)_n, P) = W(1_{[0,n+1]}, P) \]
in $M$. Equip $W(\Delta^1_M, P)$ with an augmentation
\[ W(\Delta^1_M, P) \to W(1, P) \cong P \]
via the $G$-prop map
\[ W(1_{[0,1]}, P) \to W(1, P) \]
induced by the change-of-segment map $(\text{Id,Id}) : 1_{[0,1]} \to 1.
To make the simplicial structure maps of $W(\Delta^1_M, P)$ explicit, we will use the following notation.

**Definition 9.4.8.** Suppose $G \in \mathcal{G}$ is a graph and $f : |G| \rightarrow \{0, \ldots, n + 1\}$ is a function for some $n \geq 0$. Define the map $f_*$ as the composite

$$f_* : \mathbb{I} \xrightarrow{\mathbb{I}} [0, n+1] \xrightarrow{[0, n+1][f]} [0, n+1][G]$$

in which $\{0, \ldots, n + 1\}^{\mathcal{G}}$ is the set of functions $|G| \rightarrow \{0, \ldots, n + 1\}$.

**Lemma 9.4.9.** Suppose $P$ is a $\mathcal{G}$-prop in $\mathcal{M}$, and $0 \leq i \leq n$.

1. The face map $d_i$ on $W(\Delta^1_M, P)_n$ is uniquely determined by the commutative diagrams

$$W(\Delta^1_M, P)_{n-1}(\xi) \xleftarrow{\omega_G} (\mathbb{I}_{[0, n+1]} \otimes P)[G] \xrightarrow{\sigma_i} P[G]$$

for pairs $(\xi, \sigma_i)$ of $\mathcal{C}$-profiles, $G \in \mathcal{G}(\xi)$, and functions $f : |G| \rightarrow \{0, \ldots, n + 1\}$.

2. The degeneracy map $s_i$ on $W(\Delta^1_M, P)_n$ is uniquely determined by the commutative diagrams

$$W(\Delta^1_M, P)_{n+1}(\xi) \xleftarrow{\omega_G} (\mathbb{I}_{[0, n+2]} \otimes P)[G] \xrightarrow{\sigma_i} P[G]$$

for pairs $(\xi, \sigma_i)$ of $\mathcal{C}$-profiles, $G \in \mathcal{G}(\xi)$, and functions $f : |G| \rightarrow \{0, \ldots, n + 1\}$.

**Proof.** This follows from Lemma 9.4.2 and Lemma 9.4.5.

**9.5. Bar Resolution is Boardman-Vogt Resolution**

Recall that we are now considering the free-forgetful adjunction $\mathcal{G}$

$$\mathcal{G}^\ast \mathcal{M} \xleftarrow{\mathcal{G}^\ast \mathcal{M}} \mathcal{G}^\ast \mathcal{M}.$$  

We now check that the simplicial structure maps in the bar resolution of a $\mathcal{G}$-prop arising from this adjunction correspond to those in the $W$-construction.

Suppose $\mathcal{G}$ is a shrinkable $\mathcal{C}$-colored pasting scheme, and $P$ is a $\mathcal{G}$-prop in $\mathcal{M}$. Theorem 9.3.6 and Lemma 9.4.2 yield the canonical levelwise isomorphism

$$z \xrightarrow{\zeta} (F^G_U)^{\bullet+1}P \xrightarrow{z} W(\mathbb{I}_{[0, \bullet+1]}, P) \xrightarrow{z} W(\Delta^1_M, P).$$
of graded $G$-props in $M$. The simplicial structure maps of $(F^G U)^{n+1} P$ and $W(\Delta^1_M, P)$ are recorded in Prop. 9.1.19 and Lemma 9.4.9 respectively.

**Lemma 9.5.2.** The levelwise isomorphism $\zeta$ respects all the degeneracy maps.

**Proof.** For $0 \leq i \leq n$, by Prop. 9.1.19(2) and (9.3.8), it is enough to show that the diagram

$$P[s_i H] \cong P[\text{sub}(s_i H)] \xrightarrow{(\sigma, h)_{s_i}} (I_{[0,n+2]} \otimes P)[\text{sub}(s_i H)] \xrightarrow{\omega_{\text{sub}(s_i H)}} W(\Delta^1_M, P)_{n+1}(\bar{2})$$

is commutative for each graph $(n+1)$-simplex $H = H^{[1,n+1]} \in \Delta^{n+1}(\bar{2})$, where $h$ is the map in (9.3.9). This diagram is commutative by Lemma 9.4.9(2). \qed

**Lemma 9.5.3.** The levelwise isomorphism $\zeta$ respects all the face maps.

**Proof.** There are two cases. First, for $0 \leq i \leq n-1$, by Prop. 9.1.19(1) and (9.3.8), it is enough to show that the diagram

$$P[H] \cong P[\text{sub}(H)] \xrightarrow{h} (I_{[0,n+1]} \otimes P)[\text{sub}(H)] \xrightarrow{\omega_{\text{sub}(H)}} W(\Delta^1_M, P)_{n}(\bar{2})$$

is commutative for each graph $(n+1)$-simplex $H = H^{[1,n+1]} \in \Delta^{n+1}(\bar{2})$, where $h$ is the map in (9.3.9). This diagram is commutative by Lemma 9.4.9(1).

Next, for $n \geq 1$, to prove the commutativity of the diagram

$$(F^G U)^{n+1} P(\bar{2}) \xrightarrow{\zeta} W(\Delta^1_M, P)_{n}(\bar{2})$$

pick a graph $(n+1)$-simplex $H = H^{[1,n+1]} \in \Delta^{n+1}(\bar{2})$. By Prop. 9.1.19(1) and (9.3.8), it is enough to show that the outer diagram in

$$P[H] \cong P[\text{sub}(H)] \xrightarrow{h} (I_{[0,n+1]} \otimes P)[\text{sub}(H)] \xrightarrow{\omega_{\text{sub}(H)}} W(\Delta^1_M, P)_{n}(\bar{2})$$

is commutative. Here $g(e) = i - 1$ if $e \in |H|$.
for \(2 \leq i \leq n+1\), so
\[
g_* = \bigotimes_{i=2}^{n+1} \bigotimes_{e \in [i]} \left( \mathbb{I}_{i-1} \xrightarrow{\text{inclusion}} \mathbb{I}_{[0,n]} \right).
\]
The sub-diagram \(\mathbf{1}\) is commutative by Lemma 9.4.9(1).

For the sub-diagram \(\mathbf{2}\) first note that
\[
(\partial_n h)_* = \left( \bigotimes_{e \in [i]} \left( \mathbb{I}_0 \xrightarrow{\text{inclusion}} \mathbb{I}_{[0,n]} \right) \right) \otimes g_*.
\]

There is a map
\[
\text{sub}(H) = \left( \text{sub}(d_n H) \right) (H^i) \xrightarrow{(H^i)} \text{sub}(d_n H)
\]
in the substitution category \(\mathcal{G}(\xi)\). By the coend definition of
\[
W(\Delta^1_{M}, P)_{n-1}(\xi) = W(\{\Delta^1_{M}\}_{n-1}, P) = W(\mathbb{I}_{[0,n]}, P)(\xi),
\]
there is a commutative diagram
\[
\begin{aligned}
(\mathbb{I}_{[0,n]}[\text{sub}(d_n H)] \otimes P[\text{sub}(H)]) & \xrightarrow{\Phi_n} (\mathbb{I}_{[0,n]} \otimes P)[\text{sub}(d_n H)] \\
\mathbb{I}_{[0,n]} & \downarrow \quad \downarrow \omega_{\text{sub}(d_n H)} \\
(\mathbb{I}_{[0,n]} \otimes P)[\text{sub}(H)] & \xrightarrow{\omega_{\text{sub}(H)}} W(\Delta^1_{M}, P)_{n-1}(\xi).
\end{aligned}
\]

When restricted to the summand
\[
P[\text{sub}(H)] \xrightarrow{g_*} \mathbb{I}_{[0,n]}[\text{sub}(d_n H)] \otimes P[\text{sub}(H)],
\]
the diagram (9.5.4) yields the sub-diagram \(\mathbf{2}\), which is therefore commutative. □

Here is the main result of this section. It identifies the bar resolution of a \(\mathcal{G}\)-prop with respect to the adjunction \(F \tilde{\Rightarrow} U\) with an instance of the \(W\)-construction.

**Theorem 9.5.5.** Suppose \(\mathcal{G}\) is a shrinkable \(\mathcal{C}\)-colored pasting scheme, and \(P\) is a \(\mathcal{G}\)-prop in \(M\). Then
\[
(F\mathcal{G} U)^{**+1} P \xrightarrow{\zeta} W(\Delta^1_{M}, P)
\]
in (9.5.1) is an isomorphism of simplicial \(\mathcal{G}\)-props in \(M\) augmented over \(P\).

**Proof.** We checked in Lemma 9.5.2 and Lemma 9.5.3 that the levelwise \(G\)-prop isomorphism \(\zeta\) respects the simplicial structure maps. Respecting the augmentation over \(P\) is the assertion that the diagram
\[
\begin{array}{ccc}
F\mathcal{G} UP & \xrightarrow{\zeta} & W(\mathbb{I}_{[0,1]}, P) \\
\downarrow \text{counit} & & \downarrow \\
P & \xrightarrow{z} & W(\mathbb{I}, P)
\end{array}
\]
is commutative, which is true by the \(J = \mathbb{I}_{[0,1]}\) case of Corollary 9.3.10 □
Remark 9.5.6. For small enriched categories and operads, corresponding to the pasting schemes of unital linear graphs and unital trees, the above identification is already known [BM06] (Prop. 8.3.1). However, unlike in [BM06] (Lemma 6.3), our proof does not involve the filtration on the W-construction (Prop. 5.1.15).

9.6. Realized Bar Resolution is Boardman-Vogt Resolution

Here we observe that, when M is also closed, a suitable realization of the bar resolution of a G-prop is an instance of the W-construction in M, instead of in simplicial objects in M.

The following concept is from [BM06] (Def. A.1).

Definition 9.6.1. A cosimplicial object $C^* \in M^\Delta$ is called strong monoidal if the realization functor

$$M^\Delta^{op} \xrightarrow{[?]_{c*}} M, \quad [X]_{c*} = \int_{[n]^{op}} \Delta_n \otimes C^n$$

is unit-preserving and strong symmetric monoidal and has $C^0 \cong 1$.

Remark 9.6.2. When M is symmetric monoidal closed, the realization functor $[?]_{c*}$ is a left adjoint. Its right adjoint sends $Y \in M$ to $\text{Sing}(Y) \in M^{\Delta^+}$ with

$$\text{Sing}(Y)_n = \text{Hom}_M(C^n, Y)$$

for $n \geq 0$, where $\text{Hom}_M(C^n, ?)$ is right adjoint to $? \otimes C^n$. In particular, when M is symmetric monoidal closed, the realization functor $[?]_{c*}$ preserves all colimits.

Example 9.6.3. Each of the categories of simplicial sets, topological spaces, and symmetric spectra admits a strong monoidal cosimplicial object [BM06] (Example A.16). For instance, in simplicial sets, the cosimplicial simplicial set $\Delta^*$ is strong monoidal.

Theorem 9.6.4. Suppose $G$ is a shrinkable $\mathcal{C}$-colored pasting scheme, and $P$ is a $G$-prop in M, which is symmetric monoidal closed. For each strong monoidal cosimplicial object $C^* \in M^\Delta$, there is a canonical isomorphism

$$[(F^G U)^{**1} P]_{c*} \cong W([\Delta^1_M]_{c*}, P)$$

of G-props in M augmented over P.

Proof. On the left side, the simplicial $G$-prop $(F^G U)^{**1} P$ is regarded as a $G$-prop in $sM$. The realization $[?]_{c*}$ is then applied entrywise. The strong monoidal assumption on $C^*$ ensures the result is a $G$-prop in M. On the right side, since $\Delta^1_M$ is a commutative segment in $sM$, its realization $[\Delta^1_M]_{c*}$ is a commutative segment in M, with respect to which the W-construction of P is defined. There are canonical isomorphisms of $G$-props in M:

$$[(F^G U)^{**1} P]_{c*} \cong W([\Delta^1_M, P]_{c*})$$

$$\cong W([\Delta^1_M]_{c*}, [P]_{c*})$$

$$\cong W([\Delta^1_M]_{c*}, P).$$

The first isomorphism follows from Theorem 9.5.5. The second isomorphism is from Corollary 4.4.7 and the fact that, when M is closed, the realization functor preserves colimits because it is a left adjoint. The last isomorphism follows from the fact that P is a constant simplicial $G$-prop. □
**Corollary 9.6.5.** Suppose $\mathbf{SSet}$ is the category of simplicial sets with the standard cosimplicial simplicial set $\Delta^\bullet$. Suppose $G$ is a shrinkable $\mathcal{C}$-colored pasting scheme, and $P$ is a $G_0$-cofibrant $G$-prop in $\mathbf{SSet}$. Then the augmentation

$$|(F^G U)^{\bullet \circ} P|_{\Delta^\bullet} \xrightarrow{\sim} P$$

of the realized bar resolution is a cofibrant resolution of the $G$-prop $P$ in $\mathbf{SSet}$.

**Proof.** The realization $|\Delta^1_{\mathbf{SSet}}|_{\Delta^\bullet}$ is the commutative interval $\Delta^1$ with the usual multiplicative structure induced by the maximum operation on $\{0, 1\}$ [BM06] (Lemma A.2). By Prop. 7.2.7 $G$ is compatible with $\Delta^1$. Combining Theorem 7.3.2 and Theorem 9.6.4 we conclude that in the commutative diagram

$$
\begin{tikzcd}
(F^G U)^{\bullet \circ} P |_{\Delta^\bullet} \arrow{r}{\sim} \arrow{rd}{\eta} & W(|\Delta^1_{\mathbf{SSet}}|_{\Delta^\bullet}, P) \\
& |\Delta^1_{\mathbf{SSet}}|_{\Delta^\bullet}, P
\end{tikzcd}
$$

the realized bar resolution is a cofibrant resolution of $P$ via the augmentation $|\varepsilon|_{\Delta^\bullet}$. □

The previous Corollary applies in the following situations.

**Example 9.6.6.** Suppose $G$ is the $\mathcal{C}$-colored pasting scheme of unital linear graphs, which is shrinkable. So $G$-props in $\mathbf{SSet}$ are simplicially enriched categories with object set $\mathcal{C}$, and maps of $G$-props are object-preserving simplicial functors. There are no non-trivial $\Sigma$-actions on the hom-spaces, and the unit map $* \longrightarrow P(\varepsilon)$ is a cofibration (i.e., an inclusion) for each $\varepsilon \in \mathcal{C}$. Therefore, each simplicially enriched category $P$ with object set $\mathcal{C}$ is $G_0$-cofibrant, and the realized bar resolution $|(F^G U)^{\bullet \circ} P|_{\Delta^\bullet}$ is a cofibrant resolution of $P$.

**Example 9.6.7.** Suppose $G$ is the $\mathcal{C}$-colored pasting scheme of unital trees, which is shrinkable. So $G$-props in $\mathbf{SSet}$ are simplicially enriched $\mathcal{C}$-colored operads. As in the previous case, the unit maps are automatically cofibrations. Therefore, for each simplicially enriched $\mathcal{C}$-colored $\Sigma$-cofibrant operad $P$, the realized bar resolution $|(F^G U)^{\bullet \circ} P|_{\Delta^\bullet}$ is a cofibrant resolution of $P$. For instance:

1. The associative operad $\mathbf{As}$, whose algebras are associative monoids, is $\Sigma$-cofibrant.
2. For each $1 \leq n < \infty$, there is a $\Sigma$-cofibrant $E_n$-operad $E_n \Sigma$ [Ber96].
3. For each pasting scheme $\mathcal{A}$, there is a colored operad $O$ whose algebras are $\mathcal{A}$-props [YJ15] (Section 14.1). Each entry of $O$ is a coproduct of copies of the monoidal unit indexed by strict isomorphism classes of ordered graphs, i.e., graphs equipped with an ordering on the set of vertices. The $\Sigma$-action on $O$ simply acts on the vertex ordering, and the operad $O$ is $\Sigma$-cofibrant.

**Example 9.6.8.** Suppose $G$ is the $\mathcal{C}$-colored pasting scheme of wheeled trees (or connected wheeled graphs), which is shrinkable. So $G$-props in $\mathbf{SSet}$ are simplicially enriched $\mathcal{C}$-colored wheeled (pr)operads. As above, the unit maps are automatically cofibrations. Therefore, for each simplicially enriched $\mathcal{C}$-colored $\Sigma$-cofibrant wheeled (pr)operad $P$, the realized bar resolution $|(F^G U)^{\bullet \circ} P|_{\Delta^\bullet}$ is a cofibrant resolution of $P$.
CHAPTER 10

Relative Boardman-Vogt Construction

Fix a \( \mathcal{C} \)-colored pasting scheme \( \mathcal{G} \). Suppose \((M, \otimes, \mathbb{I})\) is a cocomplete symmetric monoidal category whose monoidal product commutes with colimits on both sides, equipped with a commutative segment \((J, \mu, 0, 1, \epsilon)\). In this chapter we construct a relative version of the \( W \)-construction that applies to a map of \( \mathcal{G} \)-props and provides a categorically and homotopically well-behaved factorization of the given map. Let us stress that our relative \( W \)-construction, when applied to the pasting scheme of unital trees (for operads), is different from the one in [BM06] (Section 7); see Remark 10.2.13.

The definition of the relative \( W \)-construction is given in Section 10.1. In Section 10.2 we show that the relative \( W \)-construction is part of a pushout that simultaneously factors the original map and the augmentation of the target. Categorical and homotopical properties of the relative \( W \)-construction are discussed in Section 10.3 and Section 10.4.

10.1. Defining the Relative Boardman-Vogt Construction

Fix a map \( g : P \to Q \) of \( \mathcal{G} \)-props in \( M \). Since the pasting scheme \( \mathcal{G} \) and the commutative segment \( J \) are fixed, we will usually suppress them from the notation, so

\[
WP = W(\mathcal{G}, J, P).
\]

Motivation 10.1.1. For a \( \mathcal{G} \)-prop \( Q \) in \( M \), a typical \((\frac{1}{2})\)-entry of \( WQ \) is defined as a coend, which can also be written as a coequalizer,

\[
WQ(\frac{1}{2}) = \int_{K \in \mathcal{G}(\frac{1}{2})} (J \otimes Q)[K] = \text{coequal} \left[ \bigcup_{(H_v) \in \mathcal{G}(\frac{1}{2})(G,K)} J[K] \otimes Q[G] \xrightarrow{\mu} \bigcup_{v \in K} (J \otimes Q)[K] \right].
\]

Here

\[
(H_v) : G = K(H_v) \to K
\]
runs through the maps in the substitution category $G_{(c)}$. Given a map $g : P \rightarrow Q$ of $G$-props in $M$, its relative $W$-construction $Wg$ is designed to fit into a commutative diagram

\[
\begin{array}{c}
\text{WP} \xrightarrow{g_*} WQ \\
\eta^P \downarrow \quad \eta^Q_0 \downarrow \\
P \xrightarrow{g_0} Wg \xrightarrow{g_*} Q \quad \eta^Q_0 \downarrow \\
\end{array}
\]

of $G$-props in $M$, in which the square is a pushout.

In particular, the map $\eta^g$ provides a simultaneous factorization of the given map $g$ and the augmentation $\eta^Q$. Moreover, a $Wg$-algebra restricts to a $P$-algebra via $g_0$ and to a $WQ$-algebra via $\eta^Q_0$. From this description, it seems natural that $Wg$ should be a quotient of $WQ$ that makes the relations coming from $P$ strict. In view of the above coequalizer description of the entries of $WQ$, we should define each entry of $Wg$ as a colimit that involves a bit more than the parallel maps that define $WQ$. The precise definition is given next. Moreover, we will show that, when $g$ is homotopically nice enough, $g_0$ is a cofibration and $\eta^g$ is a weak equivalence.

**Definition 10.1.3.** Suppose $g : P \rightarrow Q$ is a map of $G$-props in $M$, and $(\frac{c}{d})$ is a pair of $C$-profiles. Define the object $Wg_{(\frac{c}{d})} \in M$ as the colimit of the diagram

\[
\begin{array}{c}
\prod_{K \in G_{(\frac{c}{d})}} (J \otimes P)[K] \\
\bigg| \downarrow g_* \bigg| \downarrow g(\epsilon, \gamma^P) \\
\prod_{(H_v) \in G_{(\frac{c}{d})}(G, K)} J[K] \otimes Q[G] \xrightarrow{J \otimes \gamma^K_{H_v}} \prod_{K \in G_{(\frac{c}{d})}} (J \otimes Q)[K]
\end{array}
\]

where

\[(H_v) : G = K(H_v) \rightarrow K\]

runs through the maps in the substitution category $G_{(\frac{c}{d})}$. The first vertical map is the coproduct over all objects $K \in G_{(\frac{c}{d})}$ of the maps

\[J[K] \otimes P[K] \xrightarrow{g_*} (\text{id}_K \otimes g) \rightarrow J[K] \otimes Q[K].\]

The other vertical map restricted to the $K \in G_{(\frac{c}{d})}$ summand is the composite

\[
\begin{array}{c}
(J \otimes P)[K] \xrightarrow{g(\epsilon, \gamma^P)} \prod_{K \in G_{(\frac{c}{d})}} (J \otimes Q)[K] \\
\bigg| \downarrow \epsilon(K) \gamma^P_K \bigg| \\
P_{(\frac{c}{d})} = (J \otimes P)[C_{(\frac{c}{d})}] \xrightarrow{g} (J \otimes Q)[C_{(\frac{c}{d})}]
\end{array}
\]
in which \( C(\omega_2) \) is the \((\omega_2)\)-corolla.

**Remark 10.1.5.** By definition each entry of \( Wg \) is a colimit of two pairs of parallel maps with a common target, one of which defines the same entry in \( WQ \). There are natural maps

\[
(J \otimes Q)[K] \xrightarrow{\omega_K} Wg(\omega_2)
\]

for \( K \in G(\omega_2) \). So, just like \( WQ \), each entry of \( Wg \) is represented by graphs with vertices decorated by \( Q \) and with ordinary internal edges decorated by the commutative segment \( J \). However, there are more relations in \( Wg \) than in \( WQ \).

**Example 10.1.6.** Let us illustrate the definition of the relative \( W \)-construction with the \( C \)-colored pasting scheme \( G = ULin \) of unital linear graphs, \( M = Top \), and \( J \) the unit interval \( ([0,1], \ast) \) equipped with the multiplication \( a \ast b = \max\{a, b\} \). Suppose \( g : P \longrightarrow Q \) is an object-preserving functor of \( Top \)-enriched categories with object set \( C \).

Reusing the notation in Example 6.1.6 for \( c, d \in C \) the relative \( W \)-construction has the entry

\[
Wg(c) = \left( \prod_{L \in ULin} J[L] \times Q[L] \right)/\sim.
\]

This is a quotient of the space of sequences

\[
q_m \circ t_{m-1} \cdots \circ t_2 \circ q_2 \circ q_1 = \begin{array}{c}
q_1 \circ t_1 \circ q_2 \\
q_2 \circ t_2 \circ c_2 \\
\vdots \\
c_{m-2} \circ q_{m-2} \circ t_{m-2} \circ c_{m-1} \circ q_m \circ c_m
\end{array}
\]

with \( m \geq 0 \), \( \{t_j\} \in [0,1]^{x_m-1} \), \( \{q_i\} \in \prod_{i=1}^m Q(c_{i-1}, c_i) \), \( \{c_j\}_{j=1}^{m-1} \in C^{x_m-1} \), \( c_0 = c \), and \( c_m = d \). So the \( q_i \)'s are composable maps in \( Q \) such that the domain of \( q_1 \) is \( c \) and that the codomain of \( q_m \) is \( d \). The empty sequence, corresponding to \( m = 0 \), is allowed if and only if \( d = c \).

The horizontal pair of maps in (10.1.4) yields unit and composition identifications of these sequences as in Example 6.1.6. The identifications corresponding to the vertical pair of maps in (10.1.4) are as follows. The identification for the case \( m = 0 \) is already accounted for by the unit identification coming from the horizontal pair of maps in (10.1.4). Indeed, the \( m = 0 \) case identifies the empty sequence with the length 1 sequence \( \text{Id}_c \), which is already part of the unit identification as described in Example 6.1.6.

For each composable sequence of \( m \geq 1 \) maps

\[
c = c_0 \xrightarrow{p_1} c_1 \xrightarrow{p_2} \cdots \xrightarrow{p_{m-1}} c_{m-1} \xrightarrow{p_m} c_m = d
\]

in \( P \) and \( \{t_j\} \in [0,1]^{x_m-1} \), there is the identification

\[
gp_m \circ t_{m-1} \cdots \circ t_2 \circ gp_1 \sim g(p_m \cdots p_1)
\]

of a length \( m \) sequence with a length 1 sequence. On the right side, note that \( g(p_m \cdots p_1) \) is not simply regarded as an element in \( Q(c,d) \). Instead, it is a length 1 sequence, i.e., an element in \( Wg(c) \) via the natural map

\[
J[L(c,d)] \times Q[L(c,d)] \longrightarrow Wg(c)
\]

with \( L(c,d) \) the linear graph \( \xrightarrow{c} \xrightarrow{1} \xrightarrow{d} \) with one vertex.
Next we define the $\mathcal{G}$-prop structure of the relative $W$-construction. As in the $W$-construction $WQ$, the $\mathcal{G}$-prop structure on the relative $W$-construction corresponds to giving the newly created ordinary internal edges length 1.

**Definition 10.1.7.** For each graph $G \in \mathcal{G}^{(2)}$, define the map $\gamma^W_G$ by declaring that the diagram

\[
\begin{array}{ccc}
\bigotimes_{v \in G} (J \otimes Q)[H_v] & \xrightarrow{\pi} & (J \otimes Q)[G(H_v)] \\
\downarrow \otimes \omega_{H_v} & & \downarrow \omega_{G(H_v)}^g \\
(Wg)[G] = \bigotimes_{v \in G} (Wg)(v) & \xrightarrow{\gamma^W_G} & Wg^{(2)}_G
\end{array}
\]

in $\mathcal{M}$ be commutative for each $\{H_v\} \in \prod_{v \in G} \mathcal{G}(v)$. Here $\pi = \otimes E 1$ is the map in (3.5.6) with $E = |G(H_v)| \setminus \prod_{v \in G} |H_v|$ and $1 : \mathcal{I} \longrightarrow J$.

**Remark 10.1.9.** Except for notational differences, the above structure map $\gamma^W_G$ has the same definition as the structure map $\gamma^W_Q$ of the $W$-construction $WQ$ (3.5.5).

**Lemma 10.1.10.** The map $\gamma^W_G$ is well-defined.

**Proof.** Recall that each entry of $Wg$ is a colimit of two pairs of parallel maps with a common target, one of which defines the same entry in $WQ$. Since we already know $\gamma^Q_G$ (3.5.5) is well-defined, it is enough to prove the outer diagram in

\[
\begin{array}{ccc}
\bigotimes_{v \in G} (J \otimes P)[H_v] & \xrightarrow{\otimes (\epsilon[H_v], \gamma^P_{H_v})} & \bigotimes_{v \in G} P(v) \\
\downarrow \otimes g & & \downarrow \otimes g \\
(J \otimes P)[K] & \xrightarrow{(\epsilon[K], \gamma^P_K)} & (J \otimes P)[C] \\
\downarrow g & & \downarrow g \\
\bigotimes_{v \in G} (J \otimes Q)[H_v] & \xrightarrow{\otimes \omega_{H_v}^g} & Wg^{(2)}_G \\
\downarrow \otimes g & & \downarrow \omega^g_K \\
(J \otimes Q)[K] & \xrightarrow{\omega^g_K} & Wg^{(2)}_G
\end{array}
\]

is commutative, in which $K = G(H_v)$ and $C$ is the corolla $C_{(2)}$. The sub-diagram $\Box$ is commutative by the associativity of $\gamma^P$ and that $\epsilon 1 = \text{Id} 1$. The sub-diagrams $\Box$ and $\Box$ are commutative by the colimit definition of $Wg^{(2)}_G$: see the vertical pair of parallel maps in (10.1.4). The other two sub-diagrams are commutative by definition.

**Proposition 10.1.11.** For each map $g : P \longrightarrow Q$ of $\mathcal{G}$-props in $\mathcal{M}$, when equipped with the above structure map, $Wg$ is a $\mathcal{G}$-prop in $\mathcal{M}$.
PROOF. We simply reuse the proof of Lemma 3.5.16 for the associativity of \( \gamma^Wg \) and the proof of Theorem 3.5.17 for unity. \( \square \)

EXAMPLE 10.1.12. Suppose \( g: P \to Q \) is a functor between small categories with \( \text{Ob}(P) = \text{Ob}(Q) = C \) and \( g = \text{Id}_C \) on objects. Recall from Prop. 8.3.1 that there is a corresponding map

\[
P^{\text{diag}} \xrightarrow{g^{\text{diag}}} Q^{\text{diag}}
\]

of \( C \)-colored operads in \( M \) with only unary operations, where \( P^{\text{diag}} \)-algebras are \( P \)-diagrams in \( M \).

With the pasting scheme \( \text{ULIn} \) of \( C \)-colored unital linear graphs and a fixed commutative segment \( J \) in \( M \), algebras of the \( C \)-colored operad \( WQ^{\text{diag}} \) are homotopy coherent \( Q \)-diagrams in \( M \) (Def. 8.3.6). As we explained in details in Section 8.3.3, a \( WQ^{\text{diag}} \)-algebra \( X = \{ X_c \in M \}_{c \in C} \) is equipped with a structure map

\[
J_{c_1} \otimes \cdots \otimes J_{c_{n-1}} \otimes X_{c_0} \xrightarrow{X(f_1,\ldots,f_n)} X_{c_n}
\]

for each sequence of \( n \geq 1 \) composable maps

\[
c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} c_{n-1} \xrightarrow{f_n} c_n \in Q.
\]

The structure maps \( \{ X(f_1,\ldots,f_n) \} \) form a system of coherent higher homotopies for the structure maps \( X(f) \) for \( f \in Q \). They satisfy some unity, composition, and associativity conditions.

Let us unravel the structure of a \( Wg^{\text{diag}} \)-algebra. For \( c,d \in C \) recall that we defined each entry of the relative \( W \)-construction \( Wg^{\text{diag}}(\cdot) \) as a colimit:

\[
\text{colim} \left( \coprod_{(H_v)} J[K] \otimes Q^{\text{diag}}[K(H_v)] \xrightarrow{j} \coprod_{(g,c,v)} J \otimes Q^{\text{diag}}[K] \xrightarrow{g} \coprod_{(g,c,v)} (J \otimes P^{\text{diag}})[K] \right)
\]

The colimit of the left pair of parallel maps is \( WQ^{\text{diag}}(\cdot) \). Taking into account the right pair of parallel maps, a \( Wg^{\text{diag}} \)-algebra \( X \) is a \( WQ^{\text{diag}} \)-algebra (i.e., a homotopy coherent \( Q \)-diagram) that satisfies one extra condition. Namely, the diagram

\[
\begin{array}{ccc}
J_{c_1} \otimes \cdots \otimes J_{c_{n-1}} \otimes X_{c_0} & \xrightarrow{(g^{\otimes n-1},\text{Id})} & X_{g(\alpha_n,\ldots,\alpha_1)} \\
\xrightarrow{(\alpha_1,\ldots,\alpha_n)} & & \\
X_{c_0} & \xrightarrow{X(g(\alpha_n,\ldots,\alpha_1))} & X_{c_n}
\end{array}
\]

is commutative for each sequence of \( n \geq 1 \) composable maps

\[
c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} c_{n-1} \xrightarrow{\alpha_n} c_n \in P.
\]

In the right vertical map, there are the composite \( \alpha_n \cdots \alpha_1 \in P(c_0,c_n) \) and its image \( g(\alpha_n,\ldots,\alpha_1) \in Q(c_0,c_n) \). In particular, a \( Wg^{\text{diag}} \)-algebra \( X \) also has the structure of a \( P \)-diagram in \( M \), where the structure map corresponding to a map \( \alpha \in P(c,d) \) is

\[
X(g\alpha) : X_c \to X_d.
\]
10.2. Relative Boardman-Vogt Construction is a Pushout

Recall that \( g : P \rightarrow Q \) is a \( \mathcal{G} \)-prop map in \( \mathcal{M} \) for some \( \mathcal{C} \)-colored pasting scheme \( \mathcal{G} \), and a commutative segment \( (J, \mu, 0, 1, \epsilon) \) in \( \mathcal{M} \) has been fixed. Next we show that the relative \( W \)-construction \( Wg \) fits into the diagram (10.1.2) with the desired categorical properties. We begin by constructing the maps \( \eta_0^Q \) and \( g_0 \).

**Lemma 10.2.1.** There is a map
\[
\eta_0^Q : WQ \rightarrow Wg
\]
of \( \mathcal{G} \)-props that is entrywise uniquely determined by the commutative diagrams
\[
\begin{array}{ccc}
(J \otimes Q)[K] & \xrightarrow{\omega_K} & (J \otimes Q)[K] \\
\downarrow \omega_K & & \downarrow \omega_K \\
WQ(\frac{G}{2}) & \xrightarrow{\eta_0^Q} & Wg(\frac{G}{2})
\end{array}
\]
for \( K \in \mathcal{G}(\frac{G}{2}) \).

**Proof.** To see that the above commutative diagrams yield a well-defined entrywise map \( \eta_0^Q \), it suffices to observe that in the colimit definition of \( Wg(\frac{G}{2}) \) (10.1.4), the coequalizer of the horizontal parallel maps is \( WQ(\frac{G}{2}) \). To see that they form a \( \mathcal{G} \)-prop map, it suffices to observe that, in both \( WQ \) and \( Wg \), the \( \mathcal{G} \)-prop structure map \( \gamma_G \) for \( G \in \mathcal{G}(\frac{G}{2}) \) is defined by the map
\[
\otimes_{v \in G} (J \otimes Q)[H_v] \xrightarrow{\pi} (J \otimes Q)[G(H_v)]
\]
in (3.5.6) with \( \{H_v\} \in \prod_{v \in G} \mathcal{G}(v) \). \( \square \)

**Lemma 10.2.2.** There is a map
\[
g_0 : P \rightarrow Wg
\]
of \( \mathcal{G} \)-props whose typical \( (\frac{G}{2}) \)-entry is the composite
\[
P(\frac{G}{2}) \xrightarrow{g} Q(\frac{G}{2}) = (J \otimes Q)[C(\frac{G}{2})] \xrightarrow{\omega_C} Wg(\frac{G}{2}),
\]
where \( C = C(\frac{G}{2}) \) is the \( (\frac{G}{2}) \)-corolla.

**Proof.** For each \( G \in \mathcal{G}(\frac{G}{2}) \) we must show that \( g_0 \) is compatible with the \( \mathcal{G} \)-prop structure maps \( \gamma_G \), i.e., that the diagram
\[
\begin{array}{ccc}
P[G] & \xrightarrow{\otimes_{v \in G} g_0} & Wg[G] \\
\gamma_G^P & \downarrow & \gamma_G^W \\
P(\frac{G}{2}) & \xrightarrow{g_0} & Wg(\frac{G}{2})
\end{array}
\]
is commutative.
is commutative. This diagram is the outer diagram in

\[
\begin{array}{ccc}
P[G] & \xo{g} & Q[G] = \xo{J \otimes Q}[C_v] & \o{g} & Wg[G] \\
\gamma_G & \o{1[G]} & & \o{1[G]} & \\
(J \otimes P)[G] & \xo{g} & (J \otimes Q)[G] & \gamma_Wg \\
\end{array}
\]

in which \(1 : \mathbb{I} \longrightarrow J\) and each \(C_v \in \mathcal{G}(v)\) is a corolla.

The sub-diagram \([1]\) is commutative because \(\epsilon 1 = \text{Id}_1\). The sub-diagram \([2]\) is commutative by definition. The sub-diagram \([3]\) is commutative by the colimit definition of \(Wg(\zeta)\) \([10.1.4]\). The sub-diagram \([4]\) is commutative by the definition of \(\gamma_W^G\) \([10.1.8]\) with \(H_v = C_v\).

\(\square\)

**Example 10.2.3.** Consider the setting of Example 6.1.6 and Example 10.1.6 with \(\mathcal{G} = \mathcal{ULin}\) the \(\mathcal{C}\)-colored pasting scheme of unital linear graphs, \(M = \text{Top}\), \(J = ([0,1], \max)\), and \(g : P \longrightarrow Q\) an object-preserving functor of \(\text{Top}\)-enriched categories with object set \(\mathcal{C}\).

1. The map

\[\eta^Q_0 : WQ(\zeta) \longrightarrow Wg(\zeta)\]

sends a length \(m \geq 0\) sequence

\[q_m \circ t_m \circ \ldots \circ t_2 \circ q_2 \circ t_1 \circ q_1 \in WQ(\zeta)\]

to the length \(m\) sequence in \(Wg(\zeta)\) denoted by the same symbol. Here \(\{q_i\} \in \prod_{j=1}^{m} Q(c_{i-1}, c_i)\) is a composable sequence of \(m\) maps, \(\{t_j\} \in [0,1]^{x_{m-1}}\), \(\{c_j\}_{j=1}^{m} \in \mathcal{C}^{x_{m-1}}\), \(c_0 = c\), and \(c_m = d\).

2. The map

\[g_0 : P(\zeta) \longrightarrow Wg(\zeta)\]

sends \(p \in P(\zeta)\) to the length 1 sequence \(gp \in Wg(\zeta)\).

Consider a length \(m \geq 1\) sequence

\[p_m \circ t_m \circ \ldots \circ t_2 \circ p_2 \circ t_1 \circ p_1 \in WP(\zeta)\]

with \(\{p_i\} \in \prod_{i=1}^{m} P(c_{i-1}, c_i)\) a composable sequence of \(m\) maps. Then

\[\eta^Q_0 g_* \circ p_m \circ t_m \circ \ldots \circ t_2 \circ p_2 \circ t_1 \circ p_1 = g(p_m \circ t_m \circ \ldots \circ t_2 \circ p_2 \circ t_1 \circ p_1) = g_0 \circ p_m \circ t_m \circ \ldots \circ t_2 \circ p_2 \circ t_1 \circ p_1\]

in \(Wg(\zeta)\). The first equality comes from the definitions of the maps \(\eta^Q_0\) and \(g_* : WP \longrightarrow WQ\). The second equality was explained in Example 10.1.6. The third
equality comes from the definitions of the map \( g_0 \) and the augmentation \( \eta^P \). It follows that
\[ \eta^Q g_* = g_0 \eta^P : WP \longrightarrow Wg. \]
The next observation says that this equality is always true.

To check that the relative \( W \)-construction is part of a pushout, we first check that the relevant square is commutative.

**Lemma 10.2.4.** The diagram
\[
\begin{array}{ccc}
WP & \xrightarrow{g_*} & WQ \\
\downarrow{\eta^P} & & \downarrow{\eta^Q} \\
P & \xrightarrow{g_0} & Wg
\end{array}
\]
is commutative.

**Proof.** It is enough to check the commutativity of the previous diagram at a typical \((d,c)\)-entry. By the coend definition of \( WP(d,c) \), it suffices to check the commutativity of the outer diagram in
\[
\begin{array}{ccc}
(J \otimes P)[K] & \xrightarrow{\otimes K g} & (J \otimes Q)[K] \\
\downarrow{(\epsilon \otimes K)^{\delta}} & & \downarrow{\omega_K} \\
P(d,c) & \xrightarrow{\alpha} & Q(d,c) = (J \otimes Q)[C(\in \mathcal{G})] \\
\downarrow{\epsilon_K} & & \downarrow{\omega^\delta} \\
\end{array}
\]
for \( K \in \mathcal{G}(d,c) \). The left sub-diagram is commutative by the colimit definition of \( Wg(d,c) \) [10.1.4]. The right sub-diagram is commutative by the definition of \( \eta_0^Q \).

Here is the first main result of this section.

**Theorem 10.2.5.** For each map \( g : P \longrightarrow Q \) of \( \mathcal{G} \)-props in \( M \), the diagram
\[
\begin{array}{ccc}
WP & \xrightarrow{g_*} & WQ \\
\downarrow{\eta^P} & & \downarrow{\eta^Q} \\
P & \xrightarrow{g_0} & Wg
\end{array}
\]
is a pushout in \( \text{Prop}^\mathcal{G}(M) \).

**Proof.** In the previous Lemma, we checked that this diagram is commutative. Suppose given \( \mathcal{G} \)-prop maps \( \alpha \) and \( \beta \) as in
\[
\begin{array}{ccc}
WP & \xrightarrow{g_*} & WQ \\
\downarrow{\eta^P} & & \downarrow{\eta^Q} \\
P & \xrightarrow{g_0} & Wg \\
\end{array}
\]
such that \( \alpha \eta^P = \beta g_* \). We must show that there is a unique map \( \delta \) that makes the whole diagram commute.
To define \( \delta \) at a typical \((\frac{3}{2})\)-entry, recall that \( \eta_0^K \) is defined by the identity map on \((J \otimes Q)[K]\) for \( K \in \mathcal{G}(\frac{3}{2}) \). Therefore, the requirement \( \delta \eta_0^K = \beta \) forces the definition of \( \delta \) as in the commutative diagram

\[
\begin{array}{c}
(J \otimes Q)[K] \xrightarrow{\omega^K} Wg(\frac{3}{2}) \\
\downarrow \omega_K \quad \downarrow \delta \\
WQ(\frac{3}{2}) \xrightarrow{\beta} Y(\frac{3}{2})
\end{array}
\]

for \( K \in \mathcal{G}(\frac{3}{2}) \). In particular, \( \delta \) must be unique if it exists. In Lemma 10.2.7 we will check that \( \delta \) is entrywise well-defined and extends \( \alpha \). In Lemma 10.2.8 we will check that \( \delta \) is a \( \mathcal{G} \)-prop map.

**Lemma 10.2.7.** The map \( \delta \) in (10.2.6) is entrywise well-defined and extends \( \alpha \).

**Proof.** To check that \( \delta \) is entrywise well-defined, by the colimit definition of the entry \( Wg(\frac{3}{2}) \), we must check that the maps \( \delta \omega^K = \beta \omega_K \) coequalize both pairs of parallel maps in (10.1.4). The maps \( \beta \omega_K \) coequalize the horizontal pair of parallel maps in (10.1.4), the coequalizer of which is \( WQ(\frac{3}{2}) \), because \( \beta : WQ \longrightarrow Y \) is defined.

To check that the maps \( \beta \omega_K \) coequalize the vertical pair of parallel maps in (10.1.4), we must show the outer diagram in

\[
\begin{array}{c}
(J \otimes P)[K] \xrightarrow{\otimes \delta g} (J \otimes Q)[K] \xrightarrow{\omega^K} WQ(\frac{3}{2}) \\
\downarrow (\otimes \delta G)^C \quad \downarrow \beta \\
P(\frac{3}{2}) \xrightarrow{\alpha} Y(\frac{3}{2}) \\
\downarrow \alpha \quad \downarrow \beta \\
Q(\frac{3}{2}) \xrightarrow{\delta} (J \otimes Q)[C] \xrightarrow{\omega^C} WQ(\frac{3}{2})
\end{array}
\]

is commutative for \( K \in \mathcal{G}(\frac{3}{2}) \), where \( C = C_{(\frac{3}{2}-)} \) is the \((\frac{3}{2})\)-corolla. The top sub-diagram is commutative by \( \omega^K \eta_p = \alpha^p g \), restricted to \((J \otimes P)[K]\). The bottom sub-diagram is commutative by \( \alpha \eta_p = \alpha^p g \), restricted to \((J \otimes P)[C] = P(\frac{3}{2}) \). We have shown that the map \( \delta \) is entrywise well-defined.

To check \( \alpha = \delta g_0 \), it is enough to check the outer diagram in

\[
\begin{array}{c}
P(\frac{3}{2}) \xrightarrow{g} Q(\frac{3}{2}) = (J \otimes Q)[C] \xrightarrow{\omega^C} Wg(\frac{3}{2}) \xrightarrow{\delta} Y(\frac{3}{2}) \\
\downarrow 1 \\
(J \otimes P)[C] \xrightarrow{\omega^C} WQ(\frac{3}{2}) \xrightarrow{\beta} Y(\frac{3}{2})
\end{array}
\]

is commutative. The top sub-diagram is the definition of the map \( g_0 \). The sub-diagram \( 1 \) is commutative by \( \alpha \eta_p = \beta g_0 \), restricted to \((J \otimes P)[C] \). The sub-diagram \( 2 \) is commutative by the definition of \( \delta \) (10.2.6). \( \square \)
Lemma 10.2.8. \( \delta : Wg \rightarrow Y \) is a map of \( \mathcal{G} \)-props.

Proof. Suppose \( G \in \mathcal{G}(\mathbb{Z}) \). We must show the outer diagram in

\[
\begin{array}{ccc}
Wg[G] & \xrightarrow{\otimes} & Y[G] \\
\downarrow & \quad & \downarrow \\
(J \otimes Q)[K] & \xrightarrow{\omega_K} & WQ(\mathbb{Z})
\end{array}
\]

is commutative. Since

\( Wg[G] = \bigotimes_{v \in G} Wg(v) \)

is a colimit, it suffices to check the restriction to \( \bigotimes_{v \in G} (J \otimes Q)[H_v] \) is commutative for \( \{H_v\} \in \prod_{v \in G} G(v) \), where \( K = G(H_v) \). The top and bottom sub-diagrams are commutative by the definition of the map \( \delta \) as in (10.2.6). The left and middle sub-diagrams are commutative by the definitions of \( \gamma^W_G \) and \( \gamma^W_Q \). The right sub-diagram is commutative by the assumption that \( \beta \) is a map of \( \mathcal{G} \)-props. \( \square \)

The proof of Theorem 10.2.5 is complete.

Corollary 10.2.9. For each map \( g : P \rightarrow Q \) of \( \mathcal{G} \)-props, there is a unique \( \mathcal{G} \)-prop map \( \eta^g \) as in

\[
\begin{array}{ccc}
WP & \xrightarrow{\eta^g} & WQ \\
\downarrow \quad \downarrow & \quad & \downarrow \quad \downarrow \\
P & \xrightarrow{\eta^g} & Wg & \xrightarrow{\eta^g} & Q
\end{array}
\]

that makes the whole diagram commute.

Example 10.2.11. As indicated in (10.2.6), the map \( \eta^g \) is uniquely determined by the commutative diagrams

\[
\begin{array}{ccc}
(J \otimes Q)[K] & \xrightarrow{\omega_K} & Wg(\mathbb{Z}) \\
\downarrow & \quad & \downarrow \\
WQ(\mathbb{Z}) & \xrightarrow{\eta^g} & Q(\mathbb{Z})
\end{array}
\]
for $K \in \mathcal{G}(\mathcal{C})$. For instance, consider the setting of Example 10.1.6 with $\mathcal{G} = \textbf{ULin}$ the $\mathcal{C}$-colored pasting scheme of unital linear graphs, $M = \textbf{Top}$, $J = ([0,1], \text{max})$, and $g : P \rightarrow Q$ an object-preserving functor of $\textbf{Top}$-enriched categories with object set $\mathcal{C}$. Then

$$\eta^P(t_{j} \cdots t_{0}) = q_{m-1} \circ t_{1} \circ q_{1} \in Q.$$ 

Here $\{q_{i}\} \in \prod_{m} q_{i}(c_{i-1}, c_{i})$ is a composable sequence of $m$ maps, $\{t_{j}\} \in [0,1]^{m-1}$, $\{c_{j}\}_{j=1}^{m} \in \mathcal{C}^{m-1}$, $c_{0} = c$, and $c_{m} = d$.

Remark 10.2.13. In [BM06] (Theorem 7.1) they defined a relative Boardman-Vogt construction of a map $g$ of operads. However, when restricted to the pasting scheme of unital trees, our relative $W$-construction $Wg$ is different from the one in [BM06], which divides out more than the pushout $Wg$. Another difference is that, just like our coend definition of $W(\mathcal{G}, J, P)$, our construction of $Wg$ is not inductive and does not involve a filtration.

The next two examples illustrate that the relative $W$-construction contains both the $W$-construction and the identity functor as special cases.

Example 10.2.14. If $P = \emptyset_{\mathcal{G}}$ is the initial $\mathcal{G}$-prop in $M$, then

$$W(\emptyset_{\mathcal{G}} \rightarrow Q) = WQ$$

for each $\mathcal{G}$-prop $Q$ in $M$, and $\eta^{Q}_{\emptyset}$ is the identity map of $WQ$. Indeed, we have that $W\emptyset_{\mathcal{G}} = \emptyset_{\mathcal{G}}$, and the augmentation $\eta^{Q}_{\emptyset}$ is the identity map. So in the pushout (10.2.10), the right vertical map $\eta^{Q}_{\emptyset}$ must also be the identity map.

Example 10.2.15. For each $\mathcal{G}$-prop $P$ in $M$, we have that

$$W(Id_{P}) = P,$$

and $g_{0} = Id_{P}$. Indeed, in the pushout (10.2.10), if $g$ is the identity map, then so are $g_{*}$ and its pushout $g_{0}$.

10.3. Naturality

The following result says that the relative $W$-construction is natural with respect to a map of maps.

Corollary 10.3.1. Suppose given a commutative diagram

$$
\begin{array}{c}
P \xrightarrow{g} Q \\
\downarrow f \\
P' \xrightarrow{g'} Q'
\end{array}
$$

of $\mathcal{G}$-props in $M$. Then there is a canonical commutative diagram

$$
\begin{array}{c}
P \xrightarrow{g} Wg \xrightarrow{\eta} Q \\
\downarrow f \\
P' \xrightarrow{g'} Wg' \xrightarrow{\eta'} Q'
\end{array}
$$

of $\mathcal{G}$-props in $M$. 


PROOF. Using (10.2.10) twice, once for \( g \) and once for \( g' \), there is a solid-arrow commutative diagram

\[
\begin{array}{ccc}
WP & \xrightarrow{g} & WQ \\
\downarrow{\eta^p} & & \downarrow{\eta^q} \\
P & \xrightarrow{g_0} & Wg \\
\downarrow{f} & & \downarrow{h} \\
WP' & \xrightarrow{g'} & WQ' \\
\downarrow{\eta'^p} & & \downarrow{\eta'^q} \\
P' & \xrightarrow{g'_0} & Wg' \\
\end{array}
\]

of \( G \)-props in \( M \). The desired map

\[ q : Wg \longrightarrow Wg' \]

is the unique induced map from the back pushout square to the front pushout square. The commutativity of (10.3.2) is a consequence of the universal property of pushouts. \( \square \)

**Remark 10.3.3.** In the previous result, the map \( q \) is uniquely determined by the commutative diagrams

\[
\begin{array}{ccc}
Wg(\xi) & \xleftarrow{\omega'_K} & (J \otimes Q)[K] \\
\downarrow{\eta} & & \downarrow{\otimes h} \\
Wg'(\xi) & \xleftarrow{\omega'_K} & (J \otimes Q')[K] \\
\end{array}
\]

for \( K \in \mathcal{C}_2(\xi) \).

The next result shows that the relative \( W \)-construction is natural with respect to composition of maps.

**Corollary 10.3.4.** Suppose given maps \( f : P \longrightarrow Q \) and \( g : Q \longrightarrow R \) of \( G \)-props in \( M \) with \( h = gf \) the composite. Then there is a canonical commutative diagram

\[
\begin{array}{ccc}
P & \xrightarrow{f} & Q \\
\downarrow{f_0} & & \downarrow{g_0} \\
Wf & \xleftarrow{\alpha} & Wg \\
\downarrow{h_0} & & \downarrow{\beta} \\
Wh & \xleftarrow{\gamma} & Wg \\
\end{array}
\]

of \( G \)-props with the sub-diagram \( \bullet \) a pushout.
Proof. Using \([10.2.10]\) three times, once for each of \(f\), \(g\), and \(h\), there is a solid-arrow commutative diagram

\[
\begin{array}{ccc}
W_P & \xrightarrow{f_*} & W_Q \\
\downarrow{\eta_P} & & \downarrow{\eta_Q} \\
P & \xrightarrow{f_0} & Q
\end{array}
\quad
\begin{array}{ccc}
& & W_R \\
\downarrow{\eta_R} & & \downarrow{\eta_R} \\
& \xleftarrow{\beta} & \quad
\end{array}
\]

of \(G\)-props in \(M\). There is a map from the span that yields the pushout \(Wf\) to the span that yields the pushout \(Wh\), so there is a unique induced map

\[\alpha : Wf \longrightarrow Wh.\]

Likewise, there is an induced map

\[\beta : Wh \longrightarrow Wg.\]

The commutativity of the diagram \([10.3.5]\) follows from the universal property of pushouts.

The fact that \(\bullet\) is a pushout in \(\text{Prop}^\mathcal{G}(M)\) follows from a categorical diagram chasing in the previous diagram. Alternatively, we may write each of the three pushouts as

\[Wf = P \bigsqcup_{WP} WQ, \quad Wg = Q \bigsqcup_{WQ} WR, \quad Wh = P \bigsqcup_{WP} WR.\]

Then we have that

\[Q \bigsqcup_{Wf} Wh = Q \bigsqcup_{P \bigsqcup_{WP} WQ} (P \bigsqcup_{WP} WR).\]

The last object has the same universal property as the pushout \(Q \bigsqcup_{WQ} WR\), which is \(Wg\). \(\square\)

Remark 10.3.6. In the diagram \([10.3.5]\), \(\alpha\) is induced by \(g\), and \(\beta\) is induced by the identity. In other words, they are uniquely determined by the commutative diagrams

\[
\begin{array}{ccc}
(J \otimes Q)[G] & \xrightarrow{g_*} & (J \otimes R)[G] \\
\downarrow{\omega^L_\delta} & & \downarrow{\omega^R_\delta} \\
Wf(\delta) & \xrightarrow{\alpha} & Wh(\delta)
\end{array}
\quad
\begin{array}{ccc}
& & (J \otimes R)[G] \\
\downarrow{\omega^L_\delta} & & \downarrow{\omega^R_\delta} \\
W(\delta) & \xrightarrow{\beta} & Wg(\delta)
\end{array}
\]

for \(G \in \underline{\mathcal{Q}}(\delta)\).

The following observation is about the naturality of the relative \(W\)-construction with respect to a change of ambient categories. Since there is a change of categories, we will include the commutative segments in the notation.

Corollary 10.3.7. Suppose \(F : M \longrightarrow N\) is a unit-preserving lax symmetric monoidal functor, \(J\) is a commutative segment in \(M\), and \(g : P \longrightarrow Q\) is a map of
$G$-props in $\mathbf{M}$. Then there is a naturally induced commutative diagram

\[
\begin{array}{ccc}
F_P & \overset{=}{} & F_P \\
\downarrow{(Fg)_0} & & \downarrow{Fg_0} \\
W(FJ,Fg) & \overset{F_*}{} & FW(J,g) \\
\downarrow{\eta Fg} & & \downarrow{Fg_0} \\
FQ & \overset{=}{} & FW(J,g)
\end{array}
\]

of $G$-props in $\mathbf{N}$ with $F_*$ induced by $F$. Furthermore, if $F$ is strong symmetric monoidal and preserves colimits, then the map $F_*$ is an isomorphism.

**Proof.** The image $FJ$ is a commutative segment in $\mathbf{N}$. Consider the solid-arrow commutative diagram

\[
\begin{array}{ccc}
W(FJ,FP) & \overset{(Fg)_*}{} & W(FJ,FQ) \\
\downarrow{\eta FP} & & \downarrow{\eta FQ} \\
FW(J,P) & \overset{Fg_*}{} & FW(J,Q) \\
\downarrow{(Fg)_0} & & \downarrow{Fg_0} \\
FP & \overset{=}{} & FW(J,g)
\end{array}
\]

of $G$-props in $\mathbf{N}$. The front square is the $F$-image of the pushout \([10.2.10]\), which need not be a pushout of $G$-props in $\mathbf{N}$. The back square is the pushout \([10.2.10]\) applied to the map $Fg : FP \rightarrow FQ$ of $G$-props in $\mathbf{N}$. The commutativity of the left face is from Theorem \([1.4.3]\). The universal property of pushouts now yields the unique dotted arrow $F_*$ that makes the entire cube commutative. The commutativity of the diagram in the statement is also a consequence of the universal property of pushouts.

For the second assertion, assume also that $F$ is strong symmetric monoidal and preserves colimits. Recall that $G$-props in a symmetric monoidal category are algebras over a colored operad \([YJ15]\) (Theorem 14.1). The prolongation of $F$ to the $G$-prop level still preserves colimits because colimits of colored operadic algebras can be written as certain reflexive coequalizers (as in, e.g., \([EKMM97]\) (II.7.4) and \([Fre10]\) (I.4.4-I.4.6)), which are preserved by $F$. So in the previous commutative cube, the front face is also a pushout. Moreover, the two maps $F_*$ in the top face are both isomorphisms by Cor. \([4.4.7]\). Therefore, the induced map from the back pushout to the front pushout is also an isomorphism. \(\square\)
**Remark 10.3.8.** The map $F_*$ in the previous result is uniquely determined by the commutative diagrams

$$
(FJ)[K] \otimes (FQ)[K] \xrightarrow{F_1} F(J[K] \otimes Q[K])
$$

$$
\omega_K \downarrow \downarrow F_\omega_K
$$

$$
W(FJ,Fg) \xrightarrow{F_*} FW(J,g)
$$

for $K \in \mathcal{G}[\mathfrak{t}]$ and pairs $(\mathfrak{t})$ of $\mathcal{C}$-profiles, where $F_1$ is an iteration of the structure map of the monoidal functor $F$.

### 10.4. Homotopical Properties

The results in this section so far hold for a general cocomplete symmetric monoidal category $\mathcal{M}$ whose monoidal product commutes with colimits on both sides and for any pasting scheme $\mathcal{G}$. To show that the relative $W$-construction gives a cylinder object in a nice homotopical setting, we now impose some restrictions on the pasting scheme.

**Theorem 10.4.1.** Suppose $\mathcal{M}$ is a cofibrantly generated, left proper, monoidal model category with a cofibrant monoidal unit, in which the colored operad for $\mathcal{G}$-props is admissible for a $\mathcal{C}$-colored connected unital pasting scheme $\mathcal{G}$ compatible with a commutative interval $J$ in $\mathcal{M}$ (Def. 7.2.5). Suppose that for each cofibration $f : A \rightarrow B$ of $\mathcal{G}$-props in $\mathcal{M}$ with $A$ a cofibrant $\mathcal{G}$-prop, the map $f$ is entrywise a cofibration in $\mathcal{M}$. Suppose $g : P \rightarrow Q$ is a map of $\mathcal{G}$-props in $\mathcal{M}$ that is a $\mathcal{G}_0$-cofibration between $\mathcal{G}_0$-cofibrant $\mathcal{G}$-props. Then in the factorization (10.2.10)

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\mathcal{G}_0} & \mathcal{M} \\
\eta^g & \downarrow & \eta\mathcal{G} \\
\mathcal{M} & \xrightarrow{\mathcal{G}} & \mathcal{M}
\end{array}
$$

in $\mathcal{G}(\mathcal{M})$, the map $\mathcal{G}_0$ is a cofibration, and the map $\eta\mathcal{G}$ is a weak equivalence.

**Proof.** Consider the diagram

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\mathcal{G}_0} & \mathcal{M} \\
\eta^p & \downarrow & \eta\mathcal{G} \\
\mathcal{M} & \xrightarrow{\mathcal{G}} & \mathcal{M}
\end{array}
$$

in $\mathcal{G}(\mathcal{M})$ in (10.2.10), in which the square is a pushout. By Theorem 7.5.1 the map $g_*$ is a cofibration of $\mathcal{G}$-props in $\mathcal{M}$, hence so is its pushout $\eta\mathcal{G}_0$. Moreover, by Theorem 7.3.2 $\mathcal{W}P$ is a cofibrant $\mathcal{G}$-prop. So by assumption $g_*$ is entrywise a cofibration in $\mathcal{M}$.

To show that $\eta\mathcal{G}$ is a weak equivalence of $\mathcal{G}$-props, it is enough to check it entrywise in $\mathcal{M}$. Since the augmentation $\eta\mathcal{Q}$ is a weak equivalence by Theorem 7.2.17 by the 2-out-of-3 property it suffices to show $\eta_0\mathcal{G}_0$ is entrywise a weak equivalence in $\mathcal{M}$. Since the augmentation $\eta^p$ is also an entrywise weak equivalence in $\mathcal{M}$, by the left properness of $\mathcal{M}$, it is enough to note that $g_*$ is entrywise a cofibration in $\mathcal{M}$.  \qed
Example 10.4.2. If $\mathcal{M}$ is the categories of (pointed or unpointed) simplicial sets, (bounded or unbounded) chain complexes over a field $k$ of characteristics zero, simplicial $k$-modules, or small categories (with the folk model structure), then every object is cofibrant, so left properness is automatic. The compatibility of the pasting scheme $\mathcal{G}$ with the commutative interval $J$ holds by Prop. 7.2.7.

Every colored operad, in particular the one for $\mathcal{G}$-props, is admissible such that each cofibration of algebras with a cofibrant source is an entrywise cofibration in $\mathcal{M}$ [WY17] (6.2.3). So the previous Theorem applies to each map of $\mathcal{G}$-props that is an underlying $\mathcal{G}_0$-cofibration between $\mathcal{G}_0$-cofibrant $\mathcal{G}$-props.

For instance, consider the one-colored pasting scheme of unital trees (for one-colored operads) and $\mathcal{M}$ the category of simplicial sets. For each $n \geq 1$ there is a $\Sigma$-cofibration $E\Sigma_n \rightarrow E\Sigma_{n+1}$ from a $\Sigma$-cofibrant $E_n$-operad to a $\Sigma$-cofibrant $E_{n+1}$-operad [Ber96]. The same is also true if $E\Sigma_{n+1}$ is replaced by a suitable $E\infty$-operad $E\Sigma_\infty$. So Theorem 10.4.1 applies to these maps.

Corollary 10.4.3. With the same assumptions as Theorem 10.4.1, suppose $f : J \rightarrow J'$ is a map of commutative intervals in $\mathcal{M}$ with $\mathcal{G}$ compatible with both $J$ and $J'$. Then the naturally induced map

$$W(J, g) \xrightarrow{f_*} W(J', g)$$

of $\mathcal{G}$-props in $\mathcal{M}$ is a weak equivalence.

Proof. By Theorem 10.4.1 in the commutative diagram

$$\begin{array}{ccc}
W(J, g) & \xrightarrow{f_*} & W(J', g) \\
\eta^g & \sim & \sim & \eta^g \\
Q & = & Q
\end{array}$$

both vertical maps are weak equivalences. So the 2-out-of-3 property implies $f_*$ is also a weak equivalence. \qed
CHAPTER 11

Relative Bar Resolution

In Section 9 and Section 10 we studied the bar resolution of a \( \mathcal{G} \)-prop and the relative Boardman-Vogt resolution of a map of \( \mathcal{G} \)-props. In this chapter we show that these two concepts are closely related. In Theorem 9.5.5 we observed that for a shrinkable pasting scheme \( \mathcal{G} \), the bar resolution \( (F^\mathcal{G}U)^{+++1}P \) is naturally isomorphic to the \( W \)-construction \( W(\Delta^n_M, P) \) as simplicial \( \mathcal{G} \)-props augmented over \( P \). This raises the natural question of whether there is a relative version of the bar resolution that corresponds to the relative \( W \)-construction.

Relative Bar? -- Relative \( W \)

Bar Resolution -- \( W \)-Construction

In this chapter we construct a relative version of the bar resolution and observe that, for a shrinkable pasting scheme, it is an instance of the relative \( W \)-construction in simplicial objects in \( M \). See Corollary 11.3.1.

Throughout most of this chapter the setting is more general and is the same as in Section 9.1. In other words, \( M \) is a cocomplete symmetric monoidal category whose monoidal product commutes with colimits on both sides, and

\[ \mathcal{G} = (S, \mathcal{G}) \leq (S', \mathcal{G}') = \mathcal{G}' \]

is an inclusion of \( \mathcal{C} \)-colored pasting schemes. This pasting scheme inclusion yields a free-forgetful adjunction (1.7.2)

\[
\begin{array}{c}
\text{Prop}(M) \\
\Rightarrow
\end{array}
\begin{array}{c}
\text{Prop}(M)
\end{array}
\]

between \( \mathcal{G} \)-props and \( \mathcal{G}' \)-props in \( M \). In Section 9.1 for a \( \mathcal{G}' \)-prop \( P \) in \( M \), we gave an explicit description of the simplicial \( \mathcal{G}' \)-prop \( (F^\mathcal{G} \mathcal{G}' U)^{+++1}P \) augmented over \( P \), also known as the bar resolution of \( P \).

In Section 11.1 we define the relative bar resolution of a map of \( \mathcal{G} \)-props. In Section 11.2 we observe that the relative bar resolution is a pushout that simultaneously factors the original map and the augmentation of the target. In Section 11.3 we show that the relative bar resolution of a map is canonically isomorphic to the relative \( W \)-construction.

11.1. Defining the Relative Bar Resolution

Fix a map \( g : P \rightarrow Q \) of \( \mathcal{G}' \)-props in \( M \). We will frequently use the previously defined concept of a graph simplex and the category \( D^{n+1}(\mathbb{Z}) \) of graph \((n + 1)\)-simplices with profiles \((\mathbb{Z})\). The reader may wish to review the relevant Def. 9.1.5.
Motivation 11.1.1. A typical $(\frac{7}{2})$-entry of the $n$th layer $(F^G, G') U^{n+1} Q$ of the bar resolution of $Q$ is a colimit as in Theorem 9.1.14 which can be rewritten as a coequalizer:

$$(F^G, G') U^{n+1} Q(\frac{7}{2}) \cong \operatorname{colim}_{H \in D^{n+1}(\frac{7}{2})} Q[H]$$

$$\cong \operatorname{coequal} \left[ \bigsqcup_{K \in \operatorname{Mor}(D^{n+1}(\frac{7}{2}))} Q[\epsilon_K] \xrightarrow{\epsilon^{Q(K)}} \bigsqcup_{H \in D^{n+1}(\frac{7}{2})} Q[H] \right].$$

Here $K$ runs through the maps in the small category $D^{n+1}(\frac{7}{2})$ of graph $(n + 1)$-simplices with profiles $(\frac{7}{2})$. For each map $K$ in $D^{n+1}(\frac{7}{2})$,

$$Q(K): Q[\epsilon_K] \longrightarrow Q[tK]$$

is the map in (9.1.11), and $\epsilon$ is the coproduct summand inclusion.

The relative bar resolution $B \cdot g$ of $g$ is designed to fit into a commutative diagram

$$(11.1.2) \quad (F^G, G') U^{n+1} P \xrightarrow{g_*} (F^G, G') U^{n+1} Q$$

in simplicial $G'$-props in $M$, in which the square is a pushout. In particular, the map $\epsilon^g$ provides a simultaneous factorization of the given map $g$, regarded as a map between constant simplicial $G'$-props in $M$, and the augmentation $\epsilon^Q$. From this description, it seems natural that the relative bar resolution $B \cdot g$ should be a quotient of the bar resolution $(F^G, G') U^{n+1} Q$ that makes the relations coming from $P$ strict. In view of the above coequalizer description of the entries of the bar resolution of $Q$, we should define each entry of each layer of $B \cdot g$ as a colimit that involves a bit more than the parallel maps that define the bar resolution of $Q$. The precise definition will be given in Def. 11.1.4.

Definition 11.1.3. For $n \geq 0$ and a pair $(\frac{7}{2})$ of $C$-profiles, define the graph $(n + 1)$-simplex

$$C_{(\frac{7}{2})}^{[1,n+1]} = \left( C, \ldots, C \right) \in D^{n+1}(\frac{7}{2})$$

with each $C = C_{(\frac{7}{2})}$ the $(\frac{7}{2})$-corolla. If $(\frac{7}{2})$ is understood from the context, then we will omit it from the notation and write $C^{[1,n+1]}$.

Definition 11.1.4. Suppose $g: P \longrightarrow Q$ is a map of $G'$-props in $M$, and $(\frac{7}{2})$ is a pair of $C$-profiles. For $n \geq 0$ define the object $B_n g(\frac{7}{2}) \in M$ as the colimit of the
diagram
(11.1.5)
\[
\coprod_{\mathcal{H} \in \text{D}^{n+1}(\vec{2})} P[\mathcal{H}] \quad \coprod_{\mathcal{K} \in \text{Mor}(\text{D}^{n+1}(\vec{2}))} Q[\mathcal{K}]
\]
\[
\xrightarrow{i_{\gamma}^P} \quad \xrightarrow{i_{\gamma}^Q}
\]

with \( \mathcal{K} \) running through the maps in the category \( \text{D}^{n+1}(\vec{2}) \), \( \iota \) the summand inclusion, and \( Q(\mathcal{K}) \) the map in (9.1.11). The first vertical map is the coproduct over all graph \((n + 1)\)-simplices \( \mathcal{H} \in \text{D}^{n+1}(\vec{2}) \) of the maps

\[
P[\mathcal{H}] \xrightarrow{\gamma_{\mathcal{H}} = \bigotimes_{v \in \text{Mor}(\mathcal{H})} g_v} Q[\mathcal{H}].
\]

The other vertical map restricted to the \( \mathcal{H} \in \text{D}^{n+1}(\vec{2}) \) summand is the composite:

\[
P[\vec{2}] \xrightarrow{\gamma_{\vec{2}}} \coprod_{\mathcal{H} \in \text{D}^{n+1}(\vec{2})} Q[\mathcal{H}] \xrightarrow{i_{\gamma}^P} \coprod_{\mathcal{K} \in \text{Mor}(\text{D}^{n+1}(\vec{2}))} Q[\mathcal{K}]
\]

We will write

\[
Q[\mathcal{H}] \xrightarrow{\omega_{\mathcal{H}}} B_n g
\]

or simply \( \omega \) for the natural map for each \( \mathcal{H} \in \text{D}^{n+1}(\vec{2}) \).

**Remark 11.1.6.** The coequalizer of the horizontal pair of maps in (11.1.5) is isomorphic to the entry \((F^G, G')_{n+1} \mathcal{Q}(\vec{2})\) by Theorem 9.1.14. So, just like \((F^G, G')_{n+1} \mathcal{Q}(\vec{2})\), the object \( B_n g(\vec{2}) \) is a colimit of \( \mathcal{Q} \)-decorated graph \((n + 1)\)-simplices with profiles \( \vec{2} \). However, there are more relations in \( B_n g(\vec{2}) \) than in \((F^G, G')_{n+1} \mathcal{Q}(\vec{2})\).

Next we define the \( G' \)-prop structure of \( B_n g \).

**Definition 11.1.7.** In the context of Def. 11.1.4, for \( G \in G'_{\vec{2}} \) define the map \( \gamma_{B_n g} \) by the commutative diagrams

\[
\bigotimes_{v \in G} Q[H_v] \xrightarrow{\omega} Q[L] \quad \bigotimes_{v \in G} \xrightarrow{\omega_{B_n g}} \bigotimes_{v \in G} B_n g(v) \xrightarrow{\gamma_{B_n g}} B_n g(\vec{2})
\]

for

\[
\{ H_v = H_v^{[1, n+1]} \} \in \prod_{v \in G} \text{D}^{n+1}(v),
\]

where

\[
L = (H_v^{[1, n]}, G(H_v^{n+1})) \in \text{D}^{n+1}(\vec{2}).
\]
Example 11.1.10. Consider the graph \((n + 1)\)-simplex \(L\) in \((11.1.9)\).

1. If \(n = 0\), then each \(H_v \in D^1(v)\) is a graph with the same profiles as the vertex \(v \in G\), and \(L\) is the graph substitution \(G(H_v)\).

2. If \(n = 1\), then each \(H_v = (H^1_v, H^2_v) \in D^2(v)\) is a graph \(2\)-simplex with the profiles of the vertex \(v \in G\), and \(L\) is the graph \(2\)-simplex \(G(H^1_v, H^2_v)\).

3. If \(n = 2\), then each \(H_v = (H^1_v, H^2_v, H^3_v) \in D^3(v)\) is a graph \(3\)-simplex with the profiles of the vertex \(v \in G\), and \(L\) is the graph \(3\)-simplex \(G(H^1_v, H^2_v, H^3_v)\).

Remark 11.1.11. In Def. \(11.1.7\), if we replace \(B_n g\) with the \(n\)th layer \((F^G, \varphi' U)_{n+1} Q\) of the bar resolution of \(Q\), then the diagram \((11.1.8)\) is precisely how the \(G'\)-prop structure map \(\gamma_G\) is defined in the free \(G'\)-prop

\[(F^G, \varphi' U)^{n+1} Q = (F^G, \varphi' U)[(F^G, \varphi' U)^n Q]\]

relative to \(G\)-props. This is explained in \(YJ15\) (Lemma 12.6). Briefly, writing

\[R = (F^G, \varphi' U)^n Q,\]

the free \(G'\)-prop \(F^G, \varphi' U R\) is entrywise a colimit of \(R\)-decorated graphs, or equivalently a colimit of \(Q\)-decorated graph \((n+1)\)-simplices. The free \(G'\)-prop structure is given by graph substitution of \(R\)-decorated graphs. In terms of \(Q\)-decorated graph simplices, this corresponds to graph substitution of the highest indexed layer, i.e., \(G(H_v^{n+1})\) in \(L\) \((11.1.9)\).

Lemma 11.1.12. For each graph \(G \in G'(\frac{\varphi}{2})\), the map

\[\gamma^{B_n g}_G : B_n g[G] \longrightarrow B_n g(\frac{\varphi}{2})\]

in \((11.1.8)\) is well-defined.

Proof. Since each \(B_n g(v)\) is defined as a colimit \((11.1.5)\), by the commutation of the monoidal product with colimits, \((B_n g)[G]\) is also a colimit. By Remark \(11.1.6\) and Remark \(11.1.11\) it remains to show that \(\gamma^{B_n g}\) is compatible with the vertical
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So it suffices to show that the outer diagram in

\[
\begin{array}{ccc}
\otimes_{v \in G} \mathcal{P}[H_v] & \xrightarrow{\otimes_{v \in G} \gamma^P_{\text{sub}(H_v)}} & \mathcal{P}[(C^{[1,n]}_v,G)] \\
\Downarrow \gamma^P_{\text{sub}(L)} & \Downarrow 1 & \Downarrow \gamma^P_G \\
\otimes_{v \in G} \mathcal{Q}[H_v] & \xrightarrow{\otimes_{v \in G} \gamma^Q_{\text{sub}(H_v)}} & \mathcal{Q}[(C^{[1,n]}_v,G)] \\
\Downarrow \omega & \Downarrow g & \Downarrow \omega \\
\otimes_{v \in G} \mathcal{Q}[L] & \xrightarrow{\omega} & B_n g(\frac{3}{2}) \\
\Downarrow 2 & \Downarrow \gamma^Q_{\text{sub}(L)} & \Downarrow 4 \\
\mathcal{Q}[L] & \xrightarrow{\omega} & B_n g(\frac{3}{2}) \\
\end{array}
\]

is commutative, where \( L \) is as in \( \boxed{11.1.9} \). To see that the top row makes sense, note that the target of the map \( \otimes_{v \in G} \gamma^P_{\text{sub}(H_v)} \) is

\[
\otimes_{v \in G} \mathcal{P}(v) = \otimes_{v \in G} \mathcal{P}[(C^{[1,n+1]}_v,G)] = \mathcal{P}[(C^{[1,n]}_v,G)],
\]

and the same equalities hold with \( Q \) in place of \( P \). The sub-diagram \( 1 \) is commutative by the associativity of the \( G' \)-prop structure map \( \gamma^P \). The sub-diagram \( 2 \) is commutative by definition. The sub-diagrams \( 3 \) and \( 4 \) are commutative by the colimit definition of \( B_n g(\frac{3}{2}) \); see the vertical pair of maps in \( \boxed{11.1.5} \).

\[ \square \]

**Proposition 11.1.13.** For each map \( g : P \to Q \) of \( G' \)-props in \( M \) and each \( n \geq 0 \), when equipped with the structure map in \( \boxed{11.1.8} \), \( B_n g \) is a \( G' \)-prop in \( M \).

**Proof.** We must show that the structure maps \( \gamma^{B_n g} \) are unital and associative with respect to graph substitution. In the definition of \( \gamma^{B_n g} \), the graph \((n+1)\)-simplex \( L \) \( \boxed{11.1.9} \) is obtained by substituting the top layer \( \{H^{n+1}_v\} \) into \( G \), followed by the finite family \( \{H^{[1,n]}_v\} \) of graph \( n \)-simplices. If \( G \) is the corolla \( C = C_{(\omega \sqcup \emptyset)} \), then

\[
C(H^{n+1}_v) = H^{n+1}_v
\]

because corollas are two-sided units for graph substitution. So the structure map \( \gamma^{B_n g}_C \) is the identity map.

Similarly, associativity of \( \gamma^{B_n g} \) is a consequence of the associativity of graph substitution. So if instead of \( G \) we have, say \( G(K_v) \), then associativity is a consequence of the equality

\[
G[K_v(H^{n+1}_u)] = [G(K_v)](H^{n+1}_u)
\]
of iterated graph substitutions. More precisely, associativity is the assertion that the outer diagram in

\[
\begin{array}{ccccccccc}
B_n g(G(K_v)) & \otimes & B_n g(G) \\
\downarrow & & \downarrow \\
\otimes & & \otimes \\
Q[H_u] & \rightarrow & Q[L_v] \\
\downarrow & & \downarrow \\
Q[L] & = & Q[L'] \\
\downarrow & & \downarrow \\
B_n g(\ell) & \rightarrow & B_n g(\ell)
\end{array}
\]

is commutative. Since

\[
B_n g(G(K_v)) = \bigotimes_{u \in G(K_v)} B_n g(u)
\]

is a colimit, it is enough to show that the restriction to the object \( \otimes_u Q[H_u] \) is commutative for families of graph \((n+1)\)-simplices

\[
\{H_u\} \in \prod_{u \in G(K_v)} D^{n+1}(u).
\]

With

\[
L = \left( H_u^{[1,n]} , G(K_v) \left( H_u^{n+1} \right) \right) \in D^{n+1}(\ell),
\]

\[
L' = \left( H_u^{[1,n]} , G(K_v(H_u^{n+1})) \right) = \left( H_u^{n+1} \right) \in D^{n+1}(\ell),
\]

\[
L_v = \left( H_u^{[1,n]} , K_v(H_u^{n+1}) \right) \in D^{n+1}(v),
\]

every sub-diagram in the previous diagram is commutative by definition. \( \square \)

Next we define the simplicial structure maps on \( B_* g \). Not surprisingly, we will use the same formulas as in the simplicial structure maps of the bar resolution (Prop. \[9.1.19\]). Recall the graph simplices \( d_i H \) and \( s_j H \) in Def. \[9.1.17\]

**Definition 11.1.14.** Suppose \( g : P \rightarrow Q \) is a map of \( G' \)-props in \( M \) and \( \ell \) is a pair of \( C \)-profiles.

1. For \( n \geq 1 \) and \( 0 \leq i \leq n \), define the face maps \( d_i \) by the commutative diagrams

\[
\begin{array}{ccccccccc}
Q[d_n H] & \leftarrow & \otimes_{i=n}^n \rightarrow & Q[H] \\
\downarrow \omega & & \downarrow \omega \\
B_{n-1} g(\ell) & \leftarrow & B_n g(\ell) \\
\downarrow \omega & & \downarrow \omega \\
Q[d_i H] & \leftarrow & \otimes_{0 \leq i \leq n-1} \rightarrow & Q[H]
\end{array}
\]
for graph \((n+1)\)-simplices \(H \in D^{n+1}(\underline{2})\). The top horizontal map is induced by the \(G'\)-prop structure maps \(\gamma_H^p\) for \(H \in H^1\).

(2) For \(0 \leq i \leq n\) define the degeneracy maps \(s_i\) by the commutative diagrams

\[
\begin{array}{ccc}
B_n g(\underline{2}) & \xrightarrow{s_i} & B_{n+1} g(\underline{2}) \\
\omega & & \omega \\
Q[H] & \xrightarrow{z} & Q[s_i H]
\end{array}
\]

for graph \((n+1)\)-simplices \(H \in D^{n+1}(\underline{2})\).

**Lemma 11.1.15.** The face and degeneracy maps in Def. 11.1.14 are entrywise well-defined and satisfy the simplicial identities.

**Proof.** Let us first show that the face map

\[
d_n : B_n g(\underline{2}) \longrightarrow B_{n-1} g(\underline{2})
\]

is well-defined. By Remark 11.1.6 and Prop. 9.1.19 it suffices to show that \(d_n\) is compatible with the vertical pair of maps in the definition of \(B_n g(\underline{2})\) (11.1.5). So for a graph \((n+1)\)-simplex \(H = H^{[1,n+1]} \in D^{n+1}(\underline{2})\), we need to show that the outer diagram in

\[
\begin{array}{ccc}
P[H] & \xrightarrow{\gamma_{\text{sub}(H)}^p} & P(\underline{2}) \xrightarrow{g} Q(\underline{2}) = Q[C_{(\underline{2},d)}^{[1,n+1]}] \\
\Phi \gamma_H^p & & \Phi \gamma_{\text{sub}(d_n H)}^p \\
P[d_n H] & \xrightarrow{2} & Q[d_n C_{(\underline{2},d)}^{[1,n+1]}] = Q[C_{(\underline{2},d)}^{[1,n]}] \\
\Phi \gamma_{d_n H}^p & & \Phi \gamma_{d_n H}^p \\
Q[H] & \xrightarrow{\omega} & B_{n-1} g(\underline{2})
\end{array}
\]

is commutative. The upper left triangle is commutative by the associativity of the \(G'\)-prop structure map \(\gamma^p\), since

\[
d_n H = H^{[2,n+1]} \quad \text{and} \quad \text{sub}(H) = [\text{sub}(d_n H)](H^1).
\]

The sub-diagram \([1]\) is commutative because \(g\) is a map of \(G'\)-props. The sub-diagram \([2]\) is commutative by the colimit definition of \(B_{n-1} g(\underline{2})\) and the fact that \(d_n H\) is a graph \(n\)-simplex; see the vertical pair of maps in (11.1.5).

The proof that the other face maps \(d_i\) for \(0 \leq i \leq n-1\) (resp., the degeneracy maps \(s_j\)) are well-defined is essentially identical to the proof for \(d_n\). In fact, to prove the case for \(d_i\) (resp., \(s_j\)), in the previous diagram simply replace \(d_n\), \(\otimes \gamma_{H^p}\), and \(\otimes \gamma_{d_n H}^p\) with \(d_i\) (resp., \(s_j\)), \(z\), and \(\varepsilon\), respectively, and note that

\[
\text{sub}(d_i H) = \text{sub}(H) = \text{sub}(s_j H).
\]

In the case of \(s_j\), also note that

\[
s_j C_{(\underline{2},d)}^{[1,n+1]} = C_{(\underline{2},d)}^{[1,n+2]}.
\]

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Finally, to see that the $d_i$ and the $s_j$ satisfy the simplicial identities, observe that the $d_i H$ and the $s_j H$ satisfy the simplicial identities by the unity and associativity of graph substitution. For the simplicial identities that involve the last face map $d_n$, we also need to use the unity and associativity of the $G'$-prop structure maps $\gamma$. □

**Example 11.1.16.** Suppose $n \geq 2$ and $\binom{n}{2}$ is a pair of $C$-profiles. To verify the simplicial identity

$$d_{n-1} d_n = d_{n-1} d_{n-1} : B_n g(\binom{n}{2}) \to B_{n-2} g(\binom{n}{2}),$$

suppose $H = H^{[1,n+1]} \in D^{n+1}(\binom{n}{2})$. Then

$$d_{n-1} d_n H = d_{n-1} H^{[2,n+1]} = H^{[3,n+1]}$$

$$= d_{n-1}(H^2(H^1), H^{[3,n+1]}) = d_{n-1} d_{n-1} H.$$

So the required simplicial identity is a consequence of the commutative diagram

$$\begin{array}{ccc}
Q[H] & \overset{\otimes \gamma_{n+1}^{Q}}{\longrightarrow} & Q[H^{[3,n+1]}] \\
\otimes \gamma_{n+1}^{Q} & \downarrow & \\
Q[H^{[2,n+1]}] & \longrightarrow & Q[H^{[3,n+1]}]
\end{array}$$

which holds by the associativity of the $G'$-prop structure map $\gamma^Q$.

Next we check that the above simplicial structure maps respect the $G'$-prop structure maps.

**Proposition 11.1.17.** For each map $g : P \to Q$ of $G'$-props in $M$, the maps

$$d_i : B_n g \to B_{n-1} g \quad \text{for} \quad n \geq 1, 0 \leq i \leq n,$$

$$s_j : B_n g \to B_{n+1} g \quad \text{for} \quad 0 \leq j \leq n$$

in Def. 11.1.14 are $G'$-prop maps. So

$$B_* g = \{B_n g\}_{n \geq 0}$$

is a simplicial $G'$-prop in $M$.

**Proof.** Let us first show that

$$d_n : B_n g \to B_{n-1} g$$
is a map of \( \mathcal{G}' \)-props. For a graph \( G \in \mathcal{G}'(2) \), we must show the outer diagram in

is commutative for families of graph \((n + 1)\)-simplices

\[ \left\{ H_v \right\} \in \prod_{v \in G} D^{n+1}(v), \]

where

\[ L = (H_v^{[1,n]}, G(H_v^{n+1})) \in D^{n+1}(2), \]

\[ L' = d_n L = (H_v^{[2,n]}, G(H_v^{n+1})) \in D^n(2). \]

The four sub-diagrams along the boundary are commutative by the definitions of the four maps along the boundary. The middle square is commutative by definition.

As in the proof of Lemma [11.1.15], the proof that the other face maps and the degeneracy maps are \( \mathcal{G}' \)-prop maps are obtained from the previous paragraph with only cosmetic changes in notations.

\[ \square \]

### 11.2. Relative Bar Resolution is a Pushout

We continue to assume that \( \mathcal{G} \leq \mathcal{G}' \) is an inclusion of pasting schemes and \( g : P \longrightarrow Q \) is a map of \( \mathcal{G}' \)-props in \( M \). We now observe that the relative bar resolution \( B_\star g \), which is a simplicial \( \mathcal{G}' \)-prop in \( M \) (Prop. [11.1.17]), fits into the diagram [11.1.2]. We begin by constructing the maps \( \varepsilon_0^\mathcal{G} \) and \( g_0 \) in the desired pushout square.

**Lemma 11.2.1.** There is a map

\[ (F^\mathcal{G}, G')^{n+1} Q \longrightarrow B_\star g \]

of simplicial \( \mathcal{G}' \)-props in \( M \) that is levelwise defined by the commutative diagrams

\[ Q[H] \xrightarrow{\text{natural}} Q[H] \]

\[ (F^\mathcal{G}, G')^{n+1} Q(2) \xrightarrow{\varepsilon_0^\mathcal{G}} B_n g(2) \]

for graph \((n + 1)\)-simplices \( H \in D^{n+1}(2) \) for \( n \geq 0 \).
Proof. For each \( n \geq 0 \) and each pair \( (\mathcal{d}, \mathcal{c}) \) of \( \mathcal{E} \)-profiles, the map

\[
(F^{\mathcal{G}' \mathcal{U}})^{n+1} \overset{\varepsilon^{\mathcal{O}}}{\longrightarrow} B_n g(\mathcal{g})
\]

is well-defined by Remark 11.1.6. That the map \( \varepsilon^{\mathcal{O}} \) is compatible with the simplicial structure maps follows from Prop. 9.1.19 and Prop. 11.1.17. That \( \varepsilon^{\mathcal{O}} \) respects the \( \mathcal{G}' \)-prop structure maps is a consequence of Remark 11.1.11. □

As we did previously, since \( P \) is a \( \mathcal{G}' \)-prop in \( M \), we may also regard it as a constant simplicial \( \mathcal{G}' \)-prop in \( M \) with all the layers equal to \( P \) and all the simplicial structure maps equal to the identity map.

Lemma 11.2.2. There is a map

\[
P \overset{g_0}{\longrightarrow} B_{\bullet} g
\]

of simplicial \( \mathcal{G}' \)-props in \( M \) that is levelwise defined as the composite

\[
\begin{array}{ccc}
P(\mathcal{g}) & \overset{g}{\longrightarrow} & Q(\mathcal{g}) = Q[C^{[1,n+1]}_{(\mathcal{d}, \mathcal{c})}] \overset{\omega}{\longrightarrow} B_n g(\mathcal{g}) \\
\end{array}
\]

for \( n \geq 0 \) and pairs \( (\mathcal{g}, \mathcal{c}) \) of \( \mathcal{E} \)-profiles.

Proof. For each \( n \geq 0 \), first we show that

\[
g_0 : P \longrightarrow B_n g
\]

is a map of \( \mathcal{G}' \)-props. For a graph \( G \in \mathcal{G}'(\mathcal{g}) \), we must show the outer diagram in

\[
\begin{array}{ccc}
P[G] = P[L] \overset{\otimes \varepsilon^{\mathcal{O}}}{\longrightarrow} Q[G] = \bigotimes_{v \in G} Q[C^{[1,n+1]}_{v \in \mathcal{G}}] \overset{\otimes \omega}{\longrightarrow} B_n g(G) = \bigotimes_{v \in G} B_n g(v) \\
\end{array}
\]

is commutative, where

\[
L = (C^{[1,n]}_{(\mathcal{d}, \mathcal{c})}, G) \in D^{n+1}(\mathcal{g}).
\]

The sub-diagram 1 is commutative by the colimit definition of \( B_n g(\mathcal{g}) \); see the vertical pair of maps in 11.1.5. The sub-diagram 2 is commutative by the definition of the \( \mathcal{G}' \)-prop structure map \( \gamma^B_n \).
Next, to show that the map $g_0$ in the previous paragraph is compatible with the face map $d_n$, we must show the outer diagram in

$$
\begin{array}{ccc}
P(1) & \xrightarrow{g} & Q(1) = Q[C_{(\mathbb{Z}/2)}^{1,n+1}] \\
\downarrow d_n & & \downarrow \omega \\
P(0) & \xrightarrow{g_0} & Q(0) = Q[C_{(\mathbb{Z}/2)}^{1,n}]
\end{array}
$$

is commutative. The left sub-diagram is commutative by definition. The right sub-diagram is commutative by the definition of the face map (Def. 11.1.14(1))

$$d_n : B_n g(1) \longrightarrow B_{n-1} g(2)$$

because

$$d_n C_{(\mathbb{Z}/2)}^{1,n+1} = C_{(\mathbb{Z}/2)}^{1,n}$$

with $\gamma_C^Q$, the identity map for a corolla $C$.

As in the proof of Lemma 11.1.13, the proof that the map $g_0$ is compatible with the other face maps and the degeneracy maps is obtained from the previous paragraph with only cosmetic changes in notations.

**Theorem 11.2.3.** Suppose $g : P \longrightarrow Q$ is a map of $\mathcal{G}'$-props in $\mathcal{M}$. Then the diagram

$$
\begin{array}{ccc}
(F^G, G')^{*+1} P & \xrightarrow{g_*} & (F^G, G')^{*+1} Q \\
\downarrow \varepsilon^P & & \downarrow \varepsilon^Q \\
P & \xrightarrow{g_0} & B_* g
\end{array}
$$

is a pushout in simplicial $\mathcal{G}'$-props in $\mathcal{M}$.

**Proof.** To see that the above diagram is commutative, suppose $n \geq 0$ and $(\mathbb{Z}/2)$ is a pair of $\mathcal{C}$-profiles. It is enough to show the outer diagram in

$$
\begin{array}{ccc}
(F^G, G')^{n+1} P(2) & \xrightarrow{g_*} & (F^G, G')^{n+1} Q(2) \\
\downarrow \varepsilon^P & & \downarrow \varepsilon^Q \\
P[H] & \xrightarrow{g_*} & Q[H] \\
\downarrow \gamma_{h(\mathbb{Z}/2)} & & \downarrow \varepsilon^Q \\
P(2) & \xrightarrow{g} & Q(2) = Q[C_{(\mathbb{Z}/2)}^{1,n+1}] & \xrightarrow{g} & B_n g(2)
\end{array}
$$

is commutative for $H \in D^{n+1}(\mathbb{Z}/2)$, in which the unnamed maps are natural maps. The top sub-diagram is commutative by definition. The left and right sub-diagrams are commutative by the definitions of the augmentation $\varepsilon^P$ and of the map $\varepsilon^Q$. The bottom sub-diagram is commutative by the colimit definition of $B_n g(2)$; see the vertical pair of maps in 11.1.5.
11. RELATIVE BAR RESOLUTION

To check the required universal property of a pushout, suppose given a solid-arrow commutative diagram

\[
\begin{array}{ccc}
(F, G) & \rightarrow & (F, G')_U \\
g_0 & \downarrow & \downarrow \beta \\
P & \rightarrow & B \ast g \\
g_0 & \rightarrow & \delta \\
\end{array}
\]

in simplicial \(G'\)-props in \(M\). We must show that there exists a unique map \(\delta\) that makes the entire diagram commutative. By the definition of \(\varepsilon_0^Q\), the requirement \(\beta = \delta \varepsilon_0^Q\) forces the definition of \(\delta\) as in the commutative diagram

\[
\begin{array}{ccc}
Q[H] & \rightarrow & (F, G')_U^{n+1} Q[2] \\
\omega & \downarrow & \downarrow \beta \\
B \ast g[2] & \rightarrow & Y_n[2] \\
\end{array}
\]

for \(n \geq 0\) and \(H \in D^{n+1}[2]\). In particular, the uniqueness of \(\delta\), if it exists, is guaranteed.

To see that the map \(\delta\) in (11.2.4) is well-defined, first note that it is compatible with the horizontal pair of maps in the definition of \(B \ast g[2]\) because \(\beta\) does; see Remark 11.1.6. To see that \(\delta\) is compatible with the vertical pair of maps in (11.1.5), we must show the outer diagram in

\[
\begin{array}{ccc}
P[H] & \rightarrow & P[2] \\
\gamma^P \downarrow & & \downarrow \varepsilon^p \\
P[H] & \rightarrow & (F, G')_U^{n+1} P[2] \\
g_0 & \downarrow & \downarrow \gamma^P \\
Q[H] & \rightarrow & (F, G')_U^{n+1} Q[2] \\
\end{array}
\]

is commutative for \(H \in D^{n+1}[2]\), in which the unnamed maps are natural maps. The upper left square is commutative by the definition of the map \(\gamma^P\). The bottom left square is commutative by definition. The sub-diagram 1 is commutative by the
assumption that $\alpha e^P = \beta g_*$. The sub-diagram $[2]$ is the outer diagram in

\[
P_{(\xi)} = P\bigl[C_{(\xi;\delta)}^{[1,n+1],(d,\zeta)}\bigr] \xrightarrow{g} Q_{(\xi)} = Q\bigl[C_{(\xi;\delta)}^{[1,n+1],(d,\zeta)}\bigr]
\]

in which the left sub-diagram is commutative by the definition of the map $\varepsilon^P$. The top sub-diagram is commutative by definition. The bottom right square is commutative by the assumption $\alpha e^P = \beta g_*$. This shows that the map $\delta$ in (11.2.4) is well-defined.

The map $\delta$ in (11.2.4) entrywise extends $\alpha$—i.e., $\alpha = \delta g_0$—because the outer diagram in

\[
P_{(\xi)} = P\bigl[C_{(\xi;\delta)}^{[1,n+1],(d,\zeta)}\bigr] \xrightarrow{g_0} Q_{(\xi)} = Q\bigl[C_{(\xi;\delta)}^{[1,n+1],(d,\zeta)}\bigr] \xrightarrow{\gamma} B_n g_{(\xi)}
\]

is commutative for each pair $\xi$ of $\mathcal{C}$-profiles, in which the unnamed maps are natural maps. Indeed, the left and top strips are commutative by the definitions of the maps $\varepsilon^P$ and $g_0$. The sub-diagram $[1]$ is commutative by definition. The sub-diagram $[2]$ is commutative by the assumption $\alpha e^P = \beta g_*$. The sub-diagram $[3]$ is commutative by the definition of the map $\delta$.

For each $n \geq 0$, to show that

$\delta : B_n g \longrightarrow Y_n$
is a map of $G'$-props, suppose $G \in G'(\vec{2})$. We must show the outer diagram in

\[
\begin{array}{ccc}
B_n g[G] & \xrightarrow{\delta} & Y_n[G] \\
\downarrow \otimes \omega_n & & \downarrow \otimes \beta \\
\otimes Q[H] & \xrightarrow{\gamma_G^{B_n g}} & \otimes (F^G, \mathcal{G}') U^{n+1} Q(v) \\
\downarrow \zeta & & \downarrow \gamma_G^{(F^G, \mathcal{G}') U^{n+1} Q} \\
Q[L] & \xrightarrow{\delta} & (F^G, \mathcal{G}') U^{n+1} Q(\vec{2}) \\
\downarrow \omega & & \downarrow \beta \\
B_{n-1} g(\vec{2}) & \xrightarrow{\delta} & Y_{n-1}(\vec{2})
\end{array}
\]

is commutative. Since

\[B_n g[G] = \bigotimes_{v \in G} B_n g(v)\]

is a colimit, it is enough to show that the restriction to the object $\bigotimes_{v \in G} Q[H_v]$ is commutative for graph $(n + 1)$-simplices $H_v \in D^{n+1}(v)$ for $v \in G$. In the above diagram,

\[L = (H_v^{[1,n]}, G(H_v^{n+1})) \in D^{n+1}(\vec{2}),\]

and the unnamed maps are natural maps. The top and bottom sub-diagrams are commutative by the definition of the map $\delta$. The left and middle sub-diagrams are commutative by the definitions of the $G'$-prop structure maps $\gamma_G^{B_n g}$ and $\gamma_G^{(F^G, \mathcal{G}') U^{n+1} Q}$ respectively; see Remark [11.1.11] for the latter. The right sub-diagram is commutative because

\[\beta : (F^G, \mathcal{G}') U^{n+1} Q \longrightarrow Y_n\]

is a map of $G'$-props.

Finally, we show that $\delta$ is compatible with the simplicial structure maps. To show that $\delta : B_n g \longrightarrow Y_n$ is compatible with the face map $d_n$, we must show the outer diagram in

\[
\begin{array}{ccc}
B_n g(\vec{2}) & \xrightarrow{\delta} & Y_n(\vec{2}) \\
\downarrow \omega & & \downarrow \beta \\
Q[H] & \xrightarrow{\delta} & (F^G, \mathcal{G}') U^{n+1} Q(\vec{2}) \\
\downarrow d_n & & \downarrow d_n \\
Q[d_n H] & \xrightarrow{\delta} & (F^G, \mathcal{G}') U^{n} Q(\vec{2}) \\
\downarrow \omega_{d_n H} & & \downarrow \beta \\
B_{n-1} g(\vec{2}) & \xrightarrow{\delta} & Y_{n-1}(\vec{2})
\end{array}
\]

is commutative for each pair $(\vec{2})$ of $F$-profiles. Since $B_n g(\vec{2})$ is a colimit, it is enough to show that the restriction to the object $Q[H]$ is commutative for $H \in D^{n+1}(\vec{2})$. The two unnamed maps are natural maps. The top and bottom sub-diagrams are
11.3. Relative Bar Resolution is Relative Boardman-Vogt Resolution

Here we restrict to the pair $\mathcal{G}_0 \leq \mathcal{G}$ with $\mathcal{G}$ a shrinkable $\mathcal{C}$-colored pasting scheme (Def. 9.2.5) and $\mathcal{G}_0$ the sub-pasting scheme with no ordinary internal edges (Def. 4.2.1). Using this inclusion of pasting schemes, we define the categories $\mathcal{D}_{n+1}(\frac{2g}{3})$ (Def. 9.1.5) of graph simplices and the relative bar resolution $\mathbb{B}_\bullet g$ (Def. 11.1.4). The special case of Cor. 11.2.5 for the pair $\mathcal{G}_0 \leq \mathcal{G}$ provides a factorization

$$g \xrightarrow{\varepsilon^g} \mathbb{B}_\bullet g \xrightarrow{\varepsilon^Q} Q$$
of the given map. We will observe that this relative bar resolution is an instance of the relative $W$-construction.

Recall the free-forgetful adjunction (9.1.2)

$$\text{Prop}^\mathcal{G}_0(M) \xrightarrow{F^\mathcal{G}} \text{Prop}^\mathcal{G}(M).$$

Each $\mathcal{G}$-prop is automatically regarded as a constant simplicial $\mathcal{G}$-prop. Also recall the commutative segment $\Delta_1^M$ in $sM$ (Def. 9.4.1), with respect to which the relative $W$-construction $W(\Delta_1^M, g)$ is defined (Prop. 10.1.11). The special case of Cor. 10.2.9 with $J = \Delta_1^M$ provides a factorization

$$g \quad \begin{array}{c} \downarrow \quad \downarrow \\eta^g \\
P \quad \begin{array}{c} \xrightarrow{g_0} \\ \xrightarrow{\eta^g} \end{array} \quad \begin{array}{c} W(\Delta_1^M, g) \\ \xrightarrow{\eta^g} \end{array} \quad Q
\end{array}$$

of the given map.

**Corollary 11.3.1.** Suppose $\mathcal{G}$ is a shrinkable $\mathcal{C}$-colored pasting scheme and $g : P \longrightarrow Q$ is a map of $\mathcal{G}$-props in $M$. Then there is a commutative diagram

$$g \quad \begin{array}{c} \downarrow \quad \downarrow \\eta^g \\
P \quad \begin{array}{c} \xrightarrow{g_0} \\ \xrightarrow{\eta^g} \end{array} \quad \begin{array}{c} W(\Delta_1^M, g) \\ \xrightarrow{\eta^g} \end{array} \quad Q
\end{array}$$

of simplicial $\mathcal{G}$-props in $M$. Moreover, the isomorphism $\chi$ is uniquely determined by the commutative diagrams

$$g \quad \begin{array}{c} \downarrow \quad \downarrow \\eta^g \\
P \quad \begin{array}{c} \xrightarrow{g_0} \quad B_n g \\
\xrightarrow{\chi} \quad Q
\end{array} \quad \begin{array}{c} \xrightarrow{\eta^g} \\ \xrightarrow{\eta^g} \end{array} \quad \begin{array}{c} W(\Delta_1^M, g) \\
\xrightarrow{\eta^g} \quad Q
\end{array}
\end{array}$$

for $n \geq 0$, pairs $(\frac{\partial}{\partial})$ of $\mathcal{C}$-profiles, and $H \in D^{n+1}(\frac{\partial}{\partial})$, where $h$ is the map in (9.3.9).

**Proof.** The first assertion follows from Theorem 10.2.5, Theorem 9.5.5, Theorem 11.2.3 and the universal property of pushouts. The second assertion follows from Remark 9.3.7. □
CHAPTER 12

Boardman-Vogt Resolutions of Colored Cyclic Operads

In this chapter we construct the Boardman-Vogt resolution of colored cyclic operads and observe that they have nice homotopical properties. One-colored cyclic operads were introduced in [GK95]. Surveys of one-colored cyclic operads are in [Mar08] Section 6 and [MSS02] Sections 5.1 and 5.2. The Boardman-Vogt resolution of one-colored cyclic operads was described in [Luk10], following closely the approach in [BM06]. Our approach here is different and follows the earlier sections. In particular, unlike in [Luk10], each entry of our Boardman-Vogt construction of a colored cyclic operad is defined in one step as a coend indexed by a substitution category, rather than as a sequential colimit of pushouts. A close reading of earlier sections reveals that much of what we have done so far requires less than a pasting scheme. Instead, what we really need to carry out the constructions and proofs is a suitable framework of graphs and graph substitution.

In Section 12.1 we describe the relevant graphs, called unrooted trees, for colored cyclic operads. In Section 12.2 we define colored cyclic operads as algebras over a monad parametrized by unrooted trees. In Section 12.3 we discuss the Boardman-Vogt resolutions of colored cyclic operads. In Section 12.4 we prove a coherence theorem for colored cyclic operads, characterizing them in terms of just a few generating operations and generating axioms. In Section 12.5 we discuss cofibrant resolutions of chain cyclic operads and of some well-known cyclic operads constructed from moduli spaces of stable curves with marked points and of Riemann spheres with holes.

12.1. Unrooted Trees

Recall that a vertex in a graph (Def. 1.1.1) is by definition a finite set of flags.

**Definition 12.1.1.** An *unrooted tree* is a simply-connected graph $T$ (Def. 1.1.16) that:

- contains at least one leg;
- is equipped with a listing (Def. 1.2.9), i.e., an ordering on each vertex and on the set of legs.

Given a set $\mathcal{C}$, a $\mathcal{C}$-colored unrooted tree is an unrooted tree $T$ equipped with a $\mathcal{C}$-coloring (Def. 1.2.1). The set of isomorphism classes of $\mathcal{C}$-colored unrooted trees whose ordered set of legs has profile $\mathcal{C} \in \text{Prof}(\mathcal{C})$ is denoted $\text{Tree}^+(\mathcal{C})$.

As before we consider unrooted trees only up to isomorphisms. From now on, all unrooted trees are $\mathcal{C}$-colored for an arbitrary but fixed set $\mathcal{C}$. 
Example 12.1.2. For each color $c \in \mathcal{C}$, there is a $c$-colored exceptional edge

$$|c| \in \text{Tree}^+(c,c)$$

that has no vertices and consists of two flags, both of which are legs. Together these two legs form a single edge. These colored exceptional edges correspond to the colored units in $\mathcal{C}$-colored cyclic operads.

Example 12.1.3. Suppose $c = (c_1, \ldots, c_n)$ is a non-empty $\mathcal{C}$-profile. The $c$-corolla

$$C_c \in \text{Tree}^+(c)$$

is the unrooted tree with one vertex $v = \{l_1, \ldots, l_n\}$ consisting of $n$ legs such that each flag $l_i$ has color $c_i$. For both the vertex $v$ and $C_c$ itself, the leg $l_i$ is ordered by $i$. For examples, if $c = (c_1, c_2, c_3)$, then the $c$-corolla may be depicted as

where, whenever possible, we draw the flags around a vertex in a clockwise fashion according to their ordering. The structure map of a $\mathcal{C}$-colored cyclic operad corresponding to each corolla is the identity map.

Example 12.1.4. With the same setting as in the previous Example, for each permutation $\sigma \in \Sigma_n$, there is a permuted corolla

$$C_{c, \sigma} \in \text{Tree}^+(c, \sigma)$$

where $c, \sigma = (c_{\sigma(1)}, \ldots, c_{\sigma(n)})$. Its unique vertex is still $v = \{l_1, \ldots, l_n\}$ with the same ordering. But for the whole unrooted tree $C_{c, \sigma}$, the leg $l_i$ is ordered by $\sigma^{-1}(i)$. In other words, the $i$th leg in the permuted corolla $C_{c, \sigma}$ is $l_{\sigma(i)}$. For example, if $c = (c_1, c_2, c_3)$ and $\sigma = (1 \ 3 \ 2) \in \Sigma_3$, then we may visualize the permuted corolla $C_{c, \sigma} \in \text{Tree}^+(c_3, c_1, c_2)$ as:

For $\mathcal{C}$-colored cyclic operads, permuted corollas correspond to the equivariant structure.

Example 12.1.5. Since we require an unrooted tree to have at least one leg, an isolated vertex is not an unrooted tree, and neither is the leg-less graph $\text{univ}$. The following shorthand for profiles will be useful when describing grafting of unrooted trees.

Definition 12.1.6. Suppose $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_n)$ are non-empty $\mathcal{C}$-profiles, $1 \leq i \leq m$, $1 \leq j \leq n$, and $a_i = b_j$. Define the $\mathcal{C}$-profile

$$(12.1.7)\quad a_{\circ^i} b = (a_1, \ldots, a_{i-1}, b_{j+1}, \ldots, b_n, a_{i+1}, \ldots, a_m)$$

of length $m + n - 2$.

Note that the $i$th entry of $a$ and the $j$th entry of $b$ are not part of $a_{\circ^i} b$.

Example 12.1.8. For each color $a \in \mathcal{C}$, $(a)_{1 \circ^1} (a) = \emptyset$. 
Example 12.1.9. Suppose \( \underline{a} = (a_1, \ldots, a_7) \) and \( \underline{b} = (b_1, \ldots, b_6) \) with \( a_4 = b_4 \). Then
\[
\underline{a} \circ_4 \underline{b} = (a_1, a_2, b_5, b_6, b_1, b_2, b_3, a_4, a_5, a_6, a_7).
\]
The above definition is motivated by the following unrooted tree.

Example 12.1.10. With the setting of Def. [12.1.6] such that \( m + n > 2 \), we define the unrooted tree
\[
T_{i,j}^{a,b} \in \text{Tree}^+(\underline{a} \circ_i \underline{b}),
\]
called a barbell, as follows.
- Its set of flags is \( \{f_1, \ldots, f_m, g_1, \ldots, g_n\} \) with \( \iota(f_i) = g_j \), \( \iota(g_j) = f_i \), and all other flags being legs.
- The coloring is given by \( \kappa(f_k) = a_k \) for \( 1 \leq k \leq m \) and \( \kappa(g_l) = b_l \) for \( 1 \leq l \leq n \).
- There are two vertices
  \[
  u = \{f_1, \ldots, f_m\} \quad \text{and} \quad v = \{g_1, \ldots, g_n\}.
  \]
The ordering at each vertex is given by the flag subscripts.
- The ordering of the set of legs
  \[
  \{1, \ldots, m + n - 2\} \xrightarrow{\chi} \text{Leg}(T_{i,j}^{a,b}) = \{f_1, \ldots, f_m, g_1, \ldots, g_n\} \setminus \{f_i, g_j\}
  \]
is given by
\[
\chi(k) = \begin{cases} 
  f_k & \text{if } 1 \leq k \leq i - 1, \\
  g_{k+n-j+i+1} & \text{if } i \leq k \leq n-j+i-1, \\
  g_{k-n+j-i+1} & \text{if } n-j+i \leq k \leq n+i-2, \\
  f_{k-n+2} & \text{if } n+i-1 \leq k \leq m+n-2.
\end{cases}
\]
We may depict the barbell \( T_{i,j}^{a,b} \) as:

\[
\begin{array}{c}
\vdots \\
\text{u} \\
\vdots \\
\text{f}_i \\
\vdots \\
\text{g}_j \\
\vdots \\
\text{v} \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\text{u} \\
\vdots \\
\text{f}_i \\
\vdots \\
\text{g}_j \\
\vdots \\
\text{v} \\
\vdots
\end{array}
\]

To visualize the leg ordering function \( \chi \), start at the tip of the first leg \( f_1 \) and start walking clockwise around the outer boundary of the barbell. We first meet the legs \( f_2, \ldots, f_{i-1} \). Once we reach the leg \( f_{i-1} \), we cross over from the \( u \) side to the \( v \) side, and the next leg we encounter is \( g_{j+1} \). As we continue our clockwise walk, we meet \( g_{j+2}, \ldots, g_n \), followed by \( g_1, \ldots, g_{j-1} \). Then we go back to the \( u \) side and encounter \( f_{i+1}, \ldots, f_n \). This pattern explains our definition of the profile \( \underline{a} \circ_i \underline{b} \).

Recall the notion of graph substitution in Def. [1.3.4]. Colored cyclic operads will be defined as algebras over a monad whose multiplication corresponds to graph substitution of unrooted trees. By [YJ15] Prop. 6.22 unrooted trees are closed under graph substitution. An alternative formalism of unrooted trees and their graph substitution are described in [HRY∞] Section 1. We will illustrate graph substitution of unrooted trees with the following examples.
Example 12.1.11. A graph substitution involving two permuted corollas (Example 12.1.4) is a single permuted corolla. In other words, we have that
\[
[C_c\sigma\tau](C_c\sigma) = C_c(\sigma\tau)
\]
for non-empty \(c\)-profiles \(C_c\) and permutations \(\sigma, \tau \in \Sigma_{|c|}\).

Example 12.1.12. In this example we illustrate a cancellation effect of exceptional edges. Consider the following unrooted tree \(T \in \text{Tree}^+(a, b, c, H_u = C_u\text{ the }u\text{-corolla, }H_v = C_c\text{ the }c\text{-colored exceptional edge, and their graph substitution }T(C_u,|c|):}

\[
T
\]

\[
T(C_u,|c|)
\]

The graph substitution \(T(C_u,|c|)\) is the \((a, b, c)\)-corolla. In the process of graph substitution, the two legs of the exceptional edge \(|c|\) and the \(c\)-colored flag in \(u\) are all identified to yield a single \(c\)-colored leg in \(T(C_u,|c|).

Example 12.1.13. There is another cancellation effect of exceptional edges. Consider the unrooted tree \(T \in \text{Tree}^+(a, b, d, e, C_w, C_y)\) and the graph substitution \(T(C_w,|c|, C_y)\):

\[
T
\]

\[
T(C_w,|c|, C_y)
\]

The graph substitution \(T(C_w,|c|, C_y)\) is equal to the barbell
\[
T^{(a, b, c, (c, d, e))}_{3,1} \in \text{Tree}^+(a, b, d, e).
\]

In the process of graph substitution, one leg of the exceptional edge \(|c|\) is identified with the \(c\)-colored flag of \(w\), and the other leg of \(|c|\) is identified with the \(c\)-colored flag of \(y\). The two internal edges in \(T\) become one internal edge in \(T(C_w,|c|, C_y)\).

Example 12.1.14. In this example, we illustrate how general unrooted trees can be created by iterated graph substitution of the barbells in Example 12.1.10. Consider the barbells
\[
T = T^{(a, b, c, (c, d, e, f))}_{3,1} \in \text{Tree}^+(a, b, d, e, f),
\]
\[
H_u = T^{(c, d, g), (g, c, f)}_{3,1} \in \text{Tree}^+(c, d, e, f),
\]
and their graph substitution $T(C_u, H_v) \in \text{Tree}^+(a, b, d, e, f)$:

Starting with two barbells, each with one internal edge, the graph substitution has two internal edges. We may iterate this process and create an unrooted tree with $n \geq 1$ internal edges using $n$ barbells. The resulting unrooted tree may not have the prescribed leg ordering. In this case, we may substitute the result into a permuted corolla (Example 12.1.4) to obtain the correct leg ordering.

**Example 12.1.15.** In this example we illustrate that the internal edges in an unrooted tree can be created using barbells via graph substitution in any given order. Suppose

$$K = T(C_u, H_v) \in \text{Tree}^+(a, b, d, e, f)$$

is the unrooted tree in Example 12.1.14. In the presentation $T(C_u, H_v)$ of $K$, the $c$-colored internal edge is on the outside (i.e., created first in $T$), and the $g$-colored internal edge is on the inside (i.e., created next in $H_v$).

Now consider the barbells

$$S = T^{(a,b,d,g),(g,e,f)}_{4,1} \in \text{Tree}^+(a, b, d, e, f),$$

$$G_w = T^{(a,b,c),(c,d,g)}_{(3,1)} \in \text{Tree}^+(a, b, d, g),$$

and their graph substitution $S(G_w, C_y) \in \text{Tree}^+(a, b, d, e, f)$:

So we have that

$$S(G_w, C_y) = K = T(C_u, H_v).$$

In the presentation $S(G_w, C_y)$ of $K$, the $g$-colored internal edge is on the outside (i.e., created first in $S$), and the $c$-colored internal edge is on the inside (i.e., created next in $H_w$).
12.2. Colored Cyclic Operads

For a fixed set \( C \), we now define \( C \)-colored cyclic operads as algebras over a suitable monad. Recall that \( \text{Prof}(C) \) is the set of all \( C \)-profiles.

**Definition 12.2.1.** We will write \( \text{Prof}^+(C) \) for the set of all non-empty \( C \)-profiles.

**Example 12.2.2.** If \( C \) is the one-point set \( \{*\} \), then \( \text{Prof}(C) \) is the set of non-negative integers, and \( \text{Prof}^+(C) \) is the set of positive integers.

Recall that \( \text{Tree}^+(c) \) is the set of isomorphism classes of unrooted trees whose ordered set of legs has profile \( c \). For a vertex \( v \) in a \( C \)-colored unrooted tree \( T \), recall that the \( C \)-profile defined by the ordered set of flags in \( v \) is denoted by \( v \) as well.

The ambient category \( (\mathcal{M}, \otimes, 1) \) is as usual assumed to be a cocomplete symmetric monoidal category whose monoidal product commutes with colimits on both sides.

**Definition 12.2.3.** Define a monad \( (F_{\text{cyc}}, \mu, \nu) \) on \( \mathcal{M}^{\text{Prof}^+(C)} \) as follows.

**The functor:** Define the functor
\[
F_{\text{cyc}} : \mathcal{M}^{\text{Prof}^+(C)} \rightarrow \mathcal{M}^{\text{Prof}^+(C)}
\]
by
\[
F_{\text{cyc}} P(c) = \bigotimes_{T \in \text{Tree}^+(c)} P[T]
\]
for \( P \in \mathcal{M}^{\text{Prof}^+(C)} \) and \( c \in \text{Prof}^+(C) \), where
\[
P[T] = \bigotimes_{v \in T} P(v)
\]
is an unordered tensor product indexed by the set of vertices in \( T \).

**The multiplication:** For each unrooted tree \( T \), we have
\[
F_{\text{cyc}} P[T] = \bigotimes_{v \in T} F_{\text{cyc}} P(v)
\]
\[
= \bigotimes_{v \in T} \bigotimes_{H_v \in \text{Tree}^+(v)} P[H_v]
\]
\[
\cong \bigotimes_{\{H_v\} \subseteq T} P[H_v].
\]

The monadic multiplication \( \mu_P \) is defined by the commutative diagrams
\[
\begin{CD}
F_{\text{cyc}} F_{\text{cyc}} P(c) @>\cong>> F_{\text{cyc}} P[T] \\
\vdash @VV\mu_P V @VV\cong V \\
\bigotimes_{T, \{H_v\} \subseteq T} P[H_v] @<<< \bigotimes_{v \in T} P[H_v] \\
\end{CD}
\]
for \( c \in \text{Prof}^+(C) \), \( T \in \text{Tree}^+(c) \), and \( \{H_v\} \in \prod_{v \in T} \text{Tree}^+(v) \).
**The unit:** The monadic unit is defined by the corolla inclusion

\[ \mu : P(\xi) \rightarrow P[C_\xi] \xrightarrow{\text{inclusion}} F^{\text{cyc}} P(\xi) \]

in which \( C_\xi \) is the \( \xi \)-corolla in Example 12.1.3.

The proof that \((F^{\text{cyc}}, \mu, \nu)\) is actually a monad is exactly as in [YJ15] Theorem 10.38, which was written for a pasting scheme. Associativity and unity of the monad are consequences of those of graph substitution of unrooted trees.

**Example 12.2.4.** Suppose \( P \in M^{\text{Prof}^*(\xi)} \).

1. For a color \( c \in \mathcal{C} \), we have that
   \[ P[\xi_c] = 1, \]
   since the \( c \)-colored exceptional edge \( \xi_c \) (Example 12.1.2) has no vertices.

2. For \( \xi \in \text{Prof}^*(\xi) \) and \( \sigma \in \Sigma_{\xi} \), we have that
   \[ P[C_{\xi, \sigma}] = P(\xi), \]
   since the profile of the unique vertex in the permuted corolla \( C_{\xi, \sigma} \) (Example 12.1.4) is \( \xi \).

3. For the barbell \( T_{a,b}^{c,d} \) in Example 12.1.10 we have that
   \[ P[T_{a,b}^{c,d}] = P(a) \circ P(b). \]

4. For the graph
   \[ K = T(C_u, H_v) \in \text{Tree}^* (a, b, d, e, f) \]
   in Example 12.1.14 we have that
   \[ P[K] = P(u) \circ P(x) \circ P(y) = P(a, b, c) \circ P(c, d, g) \circ P(g, e, f). \]

**Definition 12.2.5.** Given a set \( \xi \), the category of algebras over the monad \((F^{\text{cyc}}, \mu, \nu)\) on \( M^{\text{Prof}^*(\xi)} \) is denoted \( M^{\text{cyc}} \). Its objects are called \( \xi \)-colored cyclic operads in \( M \).

**Remark 12.2.6.** When \( \xi \) is the one-point set, a \( \{\ast\} \)-colored cyclic operad is exactly a cyclic operad in [GK95], in which cyclic operads are defined as algebras over a monad \( T_+ \) of unrooted trees. One main difference between [GK95] and our approach is that their unrooted trees do not have an ordering on the set of flags in each vertex. So their monad \( T_+ \) is defined on a category whose objects already have an equivariant structure. In contrast, our unrooted trees come equipped with an ordering on the set of flags in each vertex. So our monad \( F^{\text{cyc}} \) is defined on the category \( M^{\text{Prof}^*(\xi)} \) in which an object is just a \( \text{Prof}^*(\xi) \)-indexed family of objects in \( M \) without any equivariant structure. In our setting, the equivariant structure comes from the permuted corollas in Example 12.1.4. If \( n \) denotes the \( \{\ast\} \)-profile of length \( n \geq 1 \), then our \( P(n) \) is denoted \( \tilde{P}(n - 1) \) in [GK95].

**Remark 12.2.7.** The one-colored cyclic operads in [MSS02] II.5.1 are slightly different from those in [GK95]. In [MSS02] cyclic operads do not have a zero component. In other words, their cyclic operads do not have the bottom entry \( P(0) (= \text{our } P(1)) \). These restricted cyclic operads are algebras over the sub-monad
Unraveling the definition of the monad $F_{\mathcal{C}}^\text{cyc}$, we have the following more explicit description of a colored cyclic operad.

**Proposition 12.2.8.** A $\mathcal{C}$-colored cyclic operad is exactly a pair $(P, \gamma^P)$ consisting of:

- an object $P \in M^{\text{Prof}^+}(\mathcal{C})$;
- a structure map

$$P[T] \xrightarrow{\gamma^P_T} P(\ell)$$

for each $\ell \in \text{Prof}^+(\mathcal{C})$ and each $T \in \text{Tree}^+(\ell)$. This data is subject to the following two conditions.

**Unity:** For each $\ell \in \text{Prof}^+(\mathcal{C})$, the structure map $\gamma^P_{C_c}$ is the identity map of $P(\ell)$, where $C_c$ is the $c$-corolla in Example 12.1.3.

**Associativity:** The diagram

$$(12.2.9) \quad \bigotimes_{v \in T} P[H_v] \xrightarrow{\otimes \gamma^P_{H_v}} \bigotimes_{v \in T} P(v) = P[T] \quad \xrightarrow{\gamma^P_T} P(T[H_v]) \xrightarrow{\gamma^P_T} P(\ell)$$

is commutative for all $\ell \in \text{Prof}^+(\mathcal{C})$, $T \in \text{Tree}^+(\ell)$, and $\{H_v\} \in \prod_{v \in T} \text{Tree}^+(v)$.

**Proof.** For an $F_{\mathcal{C}}^\text{cyc}$-algebra $P$ with structure map $\gamma : F_{\mathcal{C}}^\text{cyc} P \rightarrow P$, the structure map $\gamma^P_T$ is the composite

$$P[T] \xrightarrow{\text{inclusion}} F_{\mathcal{C}}^\text{cyc} P(\ell) \xrightarrow{\gamma^P} P(\ell)$$

for $T \in \text{Tree}^+(\ell)$. The above unity and associativity conditions then correspond to those of an $F_{\mathcal{C}}^\text{cyc}$-algebra.

**Example 12.2.10.** Suppose $P$ is a $\mathcal{C}$-colored cyclic operad.

1. For each $c \in \mathcal{C}$, the structure map

$$1 = P[1_c] \xrightarrow{\gamma^P_1} P(c,c)$$

is called the $c$-colored unit.

2. For $\ell \in \text{Prof}^+(\mathcal{C})$ and a permutation $\sigma \in \Sigma_{|\ell|}$, there is a structure map

$$P[C_{\ell,\sigma}] = P(\ell) \xrightarrow{\gamma^P_{C_{\ell,\sigma}} \sigma} P(\ell)$$

corresponding to the permuted corolla $C_{\ell,\sigma}$ in Example 12.1.4. These structure maps yield the equivariant structure on $P$. 

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$F_{\geq 2}^\text{cyc}$ of the one-colored version of $F_{\mathcal{C}}^\text{cyc}$ in which only unrooted trees with at least two legs are used. For instance, the one-legged unrooted tree $\bullet$ is not a part of the definition of $F_{\geq 2}^\text{cyc}$. 

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$\mathcal{C}$-colored cyclic operad is exactly a pair $(P, \gamma^P)$ consisting of:

- an object $P \in M^{\text{Prof}^+}(\mathcal{C})$;
- a structure map

$$P[T] \xrightarrow{\gamma^P_T} P(\ell)$$

for each $\ell \in \text{Prof}^+(\mathcal{C})$ and each $T \in \text{Tree}^+(\ell)$. This data is subject to the following two conditions.

**Unity:** For each $\ell \in \text{Prof}^+(\mathcal{C})$, the structure map $\gamma^P_{C_c}$ is the identity map of $P(\ell)$, where $C_c$ is the $c$-corolla in Example 12.1.3.

**Associativity:** The diagram

$$(12.2.9) \quad \bigotimes_{v \in T} P[H_v] \xrightarrow{\otimes \gamma^P_{H_v}} \bigotimes_{v \in T} P(v) = P[T] \quad \xrightarrow{\gamma^P_T} P(T[H_v]) \xrightarrow{\gamma^P_T} P(\ell)$$

is commutative for all $\ell \in \text{Prof}^+(\mathcal{C})$, $T \in \text{Tree}^+(\ell)$, and $\{H_v\} \in \prod_{v \in T} \text{Tree}^+(v)$.

**Proof.** For an $F_{\mathcal{C}}^\text{cyc}$-algebra $P$ with structure map $\gamma : F_{\mathcal{C}}^\text{cyc} P \rightarrow P$, the structure map $\gamma^P_T$ is the composite

$$P[T] \xrightarrow{\text{inclusion}} F_{\mathcal{C}}^\text{cyc} P(\ell) \xrightarrow{\gamma^P} P(\ell)$$

for $T \in \text{Tree}^+(\ell)$. The above unity and associativity conditions then correspond to those of an $F_{\mathcal{C}}^\text{cyc}$-algebra.

**Example 12.2.10.** Suppose $P$ is a $\mathcal{C}$-colored cyclic operad.

1. For each $c \in \mathcal{C}$, the structure map

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2. For $\ell \in \text{Prof}^+(\mathcal{C})$ and a permutation $\sigma \in \Sigma_{|\ell|}$, there is a structure map

$$P[C_{\ell,\sigma}] = P(\ell) \xrightarrow{\gamma^P_{C_{\ell,\sigma}} \sigma} P(\ell)$$

corresponding to the permuted corolla $C_{\ell,\sigma}$ in Example 12.1.4. These structure maps yield the equivariant structure on $P$. 

---

$F_{\geq 2}^\text{cyc}$ of the one-colored version of $F_{\mathcal{C}}^\text{cyc}$ in which only unrooted trees with at least two legs are used. For instance, the one-legged unrooted tree $\bullet$ is not a part of the definition of $F_{\geq 2}^\text{cyc}$.
(3) For \(a, b \in \text{Prof}^+(\mathcal{C})\), \(1 \leq i \leq |a|, 1 \leq j \leq |b|, a_i = b_j\), and \(|a| + |b| > 2\), there is a structure map

\[
P[T_{a_i b_j}] = P(a) \otimes P(b) \xrightarrow{\gamma^P_T} P(a_{\circ} b_{\circ})
\]

corresponding to the barbell in Example 12.2.10.

Next is the map version of Prop. 12.2.8.

**Proposition 12.2.11.** A map of \(\mathcal{C}\)-colored cyclic operads

\[
f : (P, \gamma^P) \rightarrow (Q, \gamma^Q)
\]

is exactly a map

\[
f : P \rightarrow Q \in M^{\text{Prof}^+(\mathcal{C})}
\]

such that the diagram

\[
\begin{array}{ccc}
P[T] & \xrightarrow{\otimes_{f}} & Q[T] \\
\downarrow_{\gamma^P_T} & & \downarrow_{\gamma^Q_T} \\
P(\zeta) & \xrightarrow{f} & Q(\zeta)
\end{array}
\]

is commutative for all \(\zeta \in \text{Prof}^+(\mathcal{C})\) and \(T \in \text{Tree}^+(\mathcal{C})\).

Recall that \(\Sigma_{\mathcal{C}}^{\text{op}}\) is the groupoid of all \(\mathcal{C}\)-profiles with right permutations \(\sigma : \zeta \rightarrow \zeta\sigma\) as isomorphisms.

**Definition 12.2.12.** Denote by \(\Sigma_{\mathcal{C}}^{\text{op}}\) the full sub-groupoid of \(\Sigma_{\mathcal{C}}^{\text{op}}\) consisting of non-empty \(\zeta\)-profiles.

**Example 12.2.13.** If \(\mathcal{C}\) is the one-point set, then the objects of \(\Sigma_{\mathcal{C}}^{\text{op}}\) are the positive integers. The group of automorphisms of the object \(n \geq 1\) is the symmetric group \(\Sigma_n\), and there are no other maps.

The next concept is the cyclic operad analogue of a \(\mathcal{G}_0\)-prop (Def. 4.2.1).

**Definition 12.2.14.** (1) Denote by \(M^{\Sigma_{\mathcal{C}}^{\text{op}}}\) the category in which an object is a pair \((X, 1)\) consisting of:

- an equivariant object \(X \in M^{\Sigma_{\mathcal{C}}^{\text{op}}}\);
- a \(c\)-colored unit \(1_c : 1 \rightarrow X(c, c)\) for each \(c \in \mathcal{C}\).

(2) Denote by

\[
\begin{array}{ccc}
M^{\Sigma_{\mathcal{C}}^{\text{op}}} & \xrightarrow{F_{\Sigma}} & M^{\text{cyc}} \\
U & \xleftarrow{\text{free-forgetful}} & U
\end{array}
\]

the free-forgetful adjunction. The right adjoint \(U\) preserves all the entries and the structure maps corresponding to exceptional edges (for the colored units) and permuted corollas (for the equivariant structure) as in Example 12.2.10.

To describe the left adjoint \(F_{\Sigma}^{\text{cyc}}\), we use the following indexing category.

**Definition 12.2.16.** For \(\zeta \in \text{Prof}^+(\mathcal{C})\) define the extension category \(E^{\text{cyc}}(\zeta)\) as follows.

- Its object set is \(\text{Tree}^+(\zeta)\).
A map has the form

\[(H_v) : T(H_v) \longrightarrow T\]

in which each \(H_v \in \text{Tree}^i(v)\) is either a permuted corolla or an exceptional edge.

- The identity of an object \(T \in \text{Tree}^i(\mathcal{C})\) is \((C_v)_{v \in T}\) with each \(C_v\) the \(v\)-corolla.
- Composition is defined by graph substitution.

Next is the cyclic operad analogue of Lemma 4.2.9. The proof is essentially identical to the pasting scheme case \([YJ15]\) (Lemmas 12.6 and 12.8).

**Lemma 12.2.17.** The left adjoint

\[F^\Sigma_{\mathcal{C}} : M^\Sigma_{\mathcal{C}} \longrightarrow M^\Sigma\]

is given entrywise by

\[F^\Sigma_{\mathcal{C}} X(\mathcal{C}) = \colim_{T \in E^\Sigma(\mathcal{C})} X[T]\]

for \(\mathcal{C} \in \text{Prof}^i(\mathcal{C})\) and \(X \in M^\Sigma_{\mathcal{C}}\).

### 12.3. Boardman-Vogt Resolution

For a fixed set \(\mathcal{C}\), we now define the Boardman-Vogt resolution of \(\mathcal{C}\)-colored cyclic operads in \((M, \otimes, 1)\) with a commutative segment \((J, \mu, 0, 1, \epsilon)\). The definitions and proofs are essentially identical to our work in earlier sections, replacing a pasting scheme \(\mathcal{G}\) and pairs of \(\mathcal{C}\)-profiles with unrooted trees and non-empty \(\mathcal{C}\)-profiles. Therefore, we will only state some of the key definitions and results and omit the detailed proofs. As before, we will write \([T]\) for the set of internal edges in \(T\) and also for its cardinality.

**Definition 12.3.1.** Suppose \(\mathcal{C} \in \text{Prof}^i(\mathcal{C})\).

1. Define the substitution category \(\text{Tree}^i(\mathcal{C})\) as having unrooted trees \(T \in \text{Tree}^i(\mathcal{C})\) as objects. A map

\[(H_v) : T(H_v) \longrightarrow T\]

in \(\text{Tree}^i(\mathcal{C})\) is a family of unrooted trees \(H_v \in \text{Tree}^i(v)\) for \(v \in T\). The identity map of \(T\) is

\[(C_v) : T = T(C_v) \longrightarrow T\]

with \(C_v\) the \(v\)-corolla. Composition is defined by graph substitution of unrooted trees.

2. Define a functor \(J : \text{Tree}^i(\mathcal{C})^{\text{op}} \longrightarrow M\) by setting

\[J[T] = \bigotimes_{e \in [T]} J = J^{\otimes [T]}\]

For a map \((H_v) : T(H_v) \longrightarrow T \in \text{Tree}^i(\mathcal{C})\), the required map

\[J[T] \longrightarrow J[T(H_v)]\]

is induced by:

- \(0 : 1 \longrightarrow J\) for each internal edge in each \(H_v\) (which must become an internal edge in \(T(H_v)\));
• the multiplication \( J \otimes J \xrightarrow{\mu} J \) for each \( H_v \) that is an exceptional edge connecting two internal edges in \( T \);
• the counit \( J \xrightarrow{\varepsilon} 1 \) for each \( H_v \) that is an exceptional edge connecting one internal edge and one leg in \( T \);
• the identity of \( 1 \) for each \( H_v \) that is exceptional edge connecting two legs in \( T \).

**Definition 12.3.2.** Suppose \( \mathcal{P} \) is a \( \mathfrak{C} \)-colored cyclic operad in \( M \), and \( \mathfrak{C} \in \text{Prof}^+(\mathfrak{C}) \).

1. Define a functor \( P : \text{Tree}^+(\mathfrak{C}) \to M \) by setting
   \[
P[T] = \bigotimes_{v \in T} P(v)
   \]
   for \( T \in \text{Tree}^+(\mathfrak{C}) \). For a map \( (H_v) : T(H_v) \to T \) in \( \text{Tree}^+(\mathfrak{C}) \), the map
   \[
P[T(H_v)] = \bigotimes_{v \in T} P[H_v] \xrightarrow{\otimes \gamma_{H_v}^P} \bigotimes_{v \in T} P(v) = P[T]
   \]
   is the tensor product of the structure maps
   \[
   \gamma_{H_v}^p : P[H_v] \to P(v).
   \]

2. Define the coend
   \[
   W^{\text{cyc}}(J, P)(\mathfrak{C}) = \int_{T \in \text{Tree}^+(\mathfrak{C})} J[T] \otimes P[T].
   \]
   The family of objects \( W^{\text{cyc}}(J, P)(\mathfrak{C}) \) as \( \mathfrak{C} \) runs through \( \text{Prof}^+(\mathfrak{C}) \) is denoted \( W^{\text{cyc}}(J, P) \).

3. For \( T \in \text{Tree}^+(\mathfrak{C}) \), define the map \( \gamma_T^{W^{\text{cyc}}(J, P)} \) by the commutative diagrams
   \[
   \begin{array}{ccc}
   \bigotimes_{v \in T} J[H_v] \otimes P[H_v] & \xrightarrow{\pi} & J[T(H_v)] \otimes P[T(H_v)] \\
   \otimes \omega_{H_v} & \downarrow & \downarrow \omega_{T(H_v)} \\
   \bigotimes_{v \in T} W^{\text{cyc}}(J, P)(v) & \xrightarrow{\gamma_T^{W^{\text{cyc}}(J, P)}} & W^{\text{cyc}}(J, P)[T] \\
   \text{Id} & \downarrow & \downarrow \\
   W^{\text{cyc}}(J, P)[T] & \xrightarrow{\gamma_T^{W^{\text{cyc}}(J, P)}} & W^{\text{cyc}}(J, P)(\mathfrak{C})
   \end{array}
   \]
   as \( \{H_v\} \) runs through \( \prod_{v \in T} \text{Tree}^+(v) \), where each \( \omega \) is the natural map.

**Theorem 12.3.3.** Suppose \( \mathcal{P} \) is a \( \mathfrak{C} \)-colored cyclic operad in \( M \).

1. When equipped with the structure maps \( \gamma^{W^{\text{cyc}}(J, P)} \), \( W^{\text{cyc}}(J, P) \) is a \( \mathfrak{C} \)-colored cyclic operad.

2. There is a natural augmentation
   \[
   \eta : W^{\text{cyc}}(J, P) \to P \in M^{\text{cyc}}
   \]
that is entrywise defined by the commutative diagrams

\[
\begin{array}{ccc}
J[T] \otimes P[T] & \xrightarrow{\otimes} & 1 \otimes T \otimes P[T] \\
\downarrow \omega_T & & \downarrow \gamma_T^P \\
W^{\text{cyc}}(J,P)(\zeta) & \xrightarrow{\eta} & P(\zeta)
\end{array}
\]

for \( \zeta \in \text{Prof}^+(\mathcal{C}) \) and \( T \in \text{Tree}^+(\zeta) \).

(3) The counit of \( P \) naturally factors as

\[
\begin{array}{ccc}
F^{\text{cyc}}_\Sigma U P & \xrightarrow{\delta} & W^{\text{cyc}}(J,P) \\
\downarrow \text{counit} & & \downarrow \eta \\
F^{\text{cyc}}_\Sigma U P(\zeta) & \xrightarrow{\delta} & W^{\text{cyc}}(J,P)(\zeta)
\end{array}
\]

with \( \delta \) uniquely determined by the commutative diagrams

\[
\begin{array}{ccc}
P[T] & \xrightarrow{1 \otimes T} & J[T] \otimes P[T] \\
\downarrow \text{natural} & & \downarrow \omega_T \\
F^{\text{cyc}}_\Sigma U P(\zeta) & \xrightarrow{\delta} & W^{\text{cyc}}(J,P)(\zeta)
\end{array}
\]

for \( \zeta \in \text{Prof}^+(\mathcal{C}) \) and \( T \in \text{Tree}^+(\zeta) \).

(4) Suppose \( M \) is a cofibrantly generated monoidal model category with a commutative interval \( J \) and a cofibrant \( 1 \) such that \( M^{\text{cyc}} \) inherits a model structure with entrywise weak equivalences and fibrations. Suppose \( P \in M^{\text{cyc}} \) such that \( UP \in M^{\text{cyc}}_{+} \) is cofibrant. Then the augmentation

\[
\eta: W^{\text{cyc}}(J,P) \longrightarrow P
\]

is a cofibrant resolution of \( P \).

**Proof.** We simply reuse the proofs of Theorem 3.5.17, Prop. 4.1.2, Theorem 4.2.14 and Theorem 7.3.2. For the unrooted tree analogue of the compatibility condition (Def. 7.2.5), observe that the unrooted tree analogue of the map \( \beta_G \) (resp., \( \beta^G_\cup \)) is an acyclic cofibration (resp., cofibration) in \( M \) by a simple induction involving the pushout product axiom and the acyclic cofibration \( 0: 1 \longrightarrow J \) (resp., cofibration \( (0,1): \mathbb{I} \cup \mathbb{I} \longrightarrow \mathbb{I} \)). The reason is that in an unrooted tree, every internal edge can be shrunk. \( \square \)

Recall the free-forgetful adjunction \( F^{\text{cyc}}_\Sigma \dashv U \) in (12.2.15). The following observation is the cyclic analogue of Theorem 9.5.3.

**Theorem 12.3.4.** Suppose \( P \) is a \( \mathcal{C} \)-colored cyclic operad in \( M \). Then there is a natural isomorphism

\[
(F^{\text{cyc}}_\Sigma U)^{**} P \xrightarrow{\simeq} W^{\text{cyc}}(\Delta^M_{+}, P)
\]

of simplicial \( \mathcal{C} \)-colored cyclic operads in \( M \) augmented over \( P \).
12.4. Coherence of Colored Cyclic Operads

Here we describe colored cyclic operads in terms of a small number of generating operations and axioms. This is not needed for the Boardman-Vogt resolution of colored cyclic operads. This material is included for completeness and future reference.

**Motivation 12.4.1.** In Prop. [12.2.8] we described a colored cyclic operad $P$ in terms of:

- the family of structure maps $\{\gamma^P_T\}$, with $T$ running through all unrooted trees;
- the unity and associativity axioms, which run through all possible graph substitution of unrooted trees.

In practice it would be convenient if we could describe a colored cyclic operad in terms of a smaller family of generating structure maps and axioms. This is indeed possible due to the combinatorial properties of unrooted trees that we encountered above.

In fact, Examples [12.1.14] and [12.1.15] illustrate the following two points.

1. Up to substituting into an outermost permuted corolla to correct the leg ordering, an unrooted tree with $n \geq 1$ internal edges can be created using $n$ barbells via iterated graph substitution with the internal edges created in any prescribed order.

2. Since transpositions generate the symmetric group on $n$ letters, any two such iterated graph substitution presentations of a given unrooted tree are connected by a finite number of elementary moves. Each elementary move interchanges the order in which two internal edges are created.

The second point is analogous to Mac Lane’s Coherence Theorem for monoidal categories, which describes monoidal categories in terms of a small number of generating operations and axioms, including the Pentagon Axiom. Therefore, one can naturally expect a coherence result for colored cyclic operads involving a small number of generating operations and axioms. We will explain the details below.

The following is our coherence result for colored cyclic operads. Its proof is adapted from the pasting scheme case [YJ15] (Ch. 6, 7, and 11).

**Theorem 12.4.2.** A $\mathcal{C}$-colored cyclic operad is exactly a tuple $(P, \circ, 1)$ with:

- an equivariant object $P \in M^{\Sigma_{+}\text{op}}$;
- a $c$-colored unit $1_c : 1 \to P(c, c)$ for each $c \in \mathcal{C}$;
- a structure map $P(a_i \circ j b) : P(a_i \circ j b) \to P(a_i \circ j b)$ with an ordered tensor product on the left, whenever $a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_n) \in \text{Prof}^\ast(\mathcal{C}), 1 \leq i \leq m, 1 \leq j \leq n, a_i = b_j$, and $m + n > 2$, with $a_i \circ j b$ as in [12.1.7].

This data is required to satisfy the following five generating axioms.
Unity: The diagram

\[
\begin{array}{ccc}
P(a) \otimes 1 & \xrightarrow{\varphi_1} & P(a) \\
P(a) \otimes P(a_i, a_i) & & \\
\downarrow (\text{Id}, 1_{a_i}) & & \\
\end{array}
\]

is commutative for all \( a = (a_1, \ldots, a_m) \in \text{Prof}'(\mathcal{C}) \) and \( 1 \leq i \leq m \).

Equivariance of units: For each \( c \in \mathcal{C} \), the diagram

\[
\begin{array}{ccc}
1 & \xrightarrow{1_c} & P(c, c) \\
\downarrow 1_c & & \downarrow (1, 2) \\
P(c, c) & & P(c, c)
\end{array}
\]

is commutative.

Commutativity of \( \circ \): The diagram

\[
\begin{array}{ccc}
P(a) \otimes P(b) & \xrightarrow{\rho_{ij}} & P(a_i \circ_j b) \\
\downarrow \text{switch} & & \downarrow \sigma \\
P(b) \otimes P(a) & \xrightarrow{\rho_{ji}} & P(b_j \circ_i a)
\end{array}
\]

is commutative whenever it is defined. The right vertical map is the equivariant structure map for the block permutation

\[
\sigma = ((1, 3)(2, 4))(j - 1, |a| - i, i - 1, |b| - j) \in \Sigma_{|a| + |b| - 2}
\]

induced by \((1, 3)(2, 4) \in \Sigma_4 \) that permutes the four consecutive blocks of the indicated lengths. This is the permutation that satisfies

\[
(a_i \circ_j b)\sigma = b_j \circ_i a.
\]

Equivariance of \( \circ \): The diagram

\[
\begin{array}{ccc}
P(a) \otimes P(b) & \xrightarrow{(\sigma, \text{Id})} & P(a \sigma) \otimes P(b) \\
\downarrow \rho_{ij} & & \downarrow \sigma_{-1(i) \rho_{ij}} \\
P(a_i \circ_j b) & \xrightarrow{\sigma'_{ij}} & P(a \sigma_{-1(i) \circ_j b})
\end{array}
\]

is commutative whenever it is defined. The bottom horizontal map is the equivariant structure map for the block permutation

\[
\sigma' = \sigma(1, \ldots, 1, |b| - 1, 1, \ldots, 1) \in \Sigma_{|a| + |b| - 2}
\]

induced by \( \sigma \in \Sigma_{|a|} \) that regards the interval \([i, i + |b| - 2]\) as a single block. This is the permutation that satisfies

\[
(a_i \circ_j b)\sigma' = a \sigma_{-1(i) \circ_j b}.
\]
**12.4. Coherence of Colored Cyclic Operads**

**Associativity of $\circ$:** The diagram

\[
\begin{array}{cc}
P(a) \otimes P(b) \otimes P(c) & \xrightarrow{(\circ_j, \text{id})} P(a) \otimes P(b \circ_j c) \\
\downarrow \text{natural} & \downarrow \circ_j \\
P(a \circ_j b) \otimes P(c) & \xrightarrow{\circ_j} P(d)
\end{array}
\]

is commutative whenever it is defined with $1 \leq j \neq k \leq |b|$, in which

\[s = \begin{cases} |b| - j + i - 1 + k & \text{if } k < j, \\ i - 1 - j + k & \text{if } k > j, \end{cases} \quad t = \begin{cases} |c| - 2 + j & \text{if } k < j, \\ j & \text{if } k > j, \end{cases}
\]

and

\[d = (a \circ_j b) \circ_k c = a \circ_l (b \circ_k c).
\]

**Proof.** Given a $\mathcal{C}$-colored cyclic operad $(P, \gamma^P)$ as in Prop. 12.2.8 its colored units $\{1_c\}$ and equivariant structure are the maps $\gamma^P_T$ when $T$ are exceptional edges $|c$ and permuted corollas $C_c\sigma$, respectively, as in Example 12.2.10 (1) and (2). The structure map $\circ_j$ is the composite

\[
P(a) \otimes P(b) \xrightarrow{\circ_j} P(a \circ_j b) \xrightarrow{\gamma^P_T} P[T\circ_j]\]

with $\gamma^P_T$ as in Example 12.2.10 (3).

The five generating axioms are consequences of:

1. corresponding facts of graph substitution involving exceptional edges, permuted corollas, and barbells;
2. associativity of $\gamma^P$.

In fact, the associativity axiom of $\circ$ with $k < j$ follows from the associativity of $\gamma^P$ for the unrooted tree

![Unrooted Tree Diagram]

and the two ways of creating it as graph substitution of barbells, as in Examples 12.1.14 and 12.1.15. For $k > j$ the associativity axiom of $\circ$ corresponds to the unrooted tree:

![Unrooted Tree Diagram]

The unity axiom is a consequence of the equality

\[T^{\#(a_i, a_i)}_{2, 1} \circ |a_i | = C_{\Omega}.
\]
which we explained in Example 12.1.12. The equivariance of units is a consequence of the equality
\[(C_{c,e})(c,e)(c) = c.\]
This is the fact that every exceptional edge is invariant under the \((1 2)\) permutation of its two legs. The commutativity of \(\circ\) follows from the equality
\[\left(C_{a,b}(a,b)\right) = T_{i,j}^{a,b}.\]
This is the fact that the barbells \(T_{i,j}^{a,b}\) only differ by a \(\sigma\)-permutation of the leg ordering.

Conversely, suppose given a tuple \((P, \circ, 1)\) satisfying the five generating axioms.

We define the structure maps
\[\gamma_P : P[T] \rightarrow P(c)\]
for \(T \in \text{Tree}(c)\) as follows. Given \(T \in \text{Tree}(c)\), we choose a presentation of \(T\) as an iterated graph substitution (12.4.3)
\[T = T_n(T_{n-1}) \cdots (T_1)\]
with each \(T_i\) a permuted corolla, an exceptional edge, or a barbell. For each \(1 \leq j \leq n - 1\), in the \(j\)th layer \(T_j\) is the only possible non-corolla; a corolla is substituted into every other vertex in \(T_{j+1}\). By the discussion in Example 12.1.14 such a presentation always exists. As forced by the associativity axiom of \(\gamma_P\) (12.2.9), we define the map \(\gamma_T^P\) as the composite (12.4.4)
\[\gamma_T^P = \gamma_{T_n} \circ \otimes \gamma_{T_{n-1}} \circ \cdots \circ \otimes \gamma_{T_1}\]
in which each \(\gamma_{T_k}^P\) is defined as:
- the equivariant structure map
  \[\sigma : P(a) \rightarrow P(a')\]
  if \(T_k\) is the permuted corolla \(C_{a,a'}\);
- the \(a\)-colored unit if \(T_k\) is the \(a\)-colored exceptional edge;
- the composite
  \[P(T_{i,j}^{a,b}) = P(a) \otimes P(b) \xrightarrow{\gamma_{T_k}} P(a \circ b)\]
  \[\circ_j\]
  \[\xrightarrow{\alpha_j} P(a) \otimes P(b)\]
  if \(T_k\) is the barbell \(T_{i,j}^{a,b}\).

Once we show that such a map \(\gamma_T^P\) is well-defined, then its unity and associativity as in Prop. 12.2.8 are guaranteed.

To see that \(\gamma_T^P\) is well-defined, we need to show that it is independent of the choice of a presentation as in (12.4.3). First, by the unity axiom and equivariance of units, we may assume that no \(T_i\) is an exceptional edge and that \(T\) itself is not an exceptional edge. Next, by the commutativity and equivariance of \(\circ\), we may
move all the permuted corollas in the presentation \((12.4.3)\) to the left. Using the equivariant structure, we compose these consecutive permuted corollas down into just one permuted corolla as in Example \((12.1.11)\). So we may assume that in our presentation of \(T\), the highest entry \(T_n\) is a permuted corolla and every entry \(T_i\) with \(i < n\) is a barbell. In particular, each \(T_i\) contains one internal edge in \(T\), and \(n - 1\) is the number of internal edges in \(T\). We call this a \textit{stratified presentation} of \(T\).

Suppose
\[
T = T_n'(T_{n-1}'\cdots(T_1')
\]
is another stratified presentation of \(T\). The two given stratified presentations are two different ways to create the internal edges in \(T\) using barbells via iterated graph substitution. By the equivariance, commutativity, and associativity of \(\circ\) and the discussion in Examples \((12.1.14)\) and \((12.1.15)\) we may assume that the internal edges in \(T\) are created in the same order with the same barbells in these two stratified presentations, i.e., \(T_i = T_i'\) for \(i < n\). Since the iterated graph substitution
\[
T_{n-1}\cdots(T_1) = T_{n-1}'\cdots(T_1')
\]
is already \(T\) except possibly for the leg ordering, the permuted corolla \(T_n\) is uniquely determined, and \(T_n = T_n'\) as well. It follows that \(\gamma_T^P\) is well-defined.

\[\square\]

\textbf{Remark 12.4.5.} In [MSS02] Remark 5.10, it was mentioned that one-colored cyclic operads can be expressed in terms of the generating operations as in Theorem \((12.4.2)\). However, the explicit generating axioms were not written down in [MSS02].

\textbf{Corollary 12.4.6.} A map \(f : P \rightarrow Q\) of \(\mathcal{C}\)-colored cyclic operads is exactly a map of equivariant objects
\[
f : P \rightarrow Q \in M^{\Sigma^+}\text{cyclic}
\]
such that:

\begin{itemize}
  \item[(1)] The diagram
  \[
  \begin{array}{ccc}
  1 & \rightarrow & P(c, c) \\
  \downarrow & & \downarrow f \\
  Q(c, c) & \rightarrow & Q(c, c)
  \end{array}
  \]
is commutative for each \(c \in \mathcal{C}\).
  \item[(2)] The diagram
  \[
  \begin{array}{ccc}
  P(a) \otimes P(b) & \rightarrow & Q(a) \otimes Q(b) \\
  \downarrow & & \downarrow \\
  P(a \circ_j b) & \rightarrow & Q(a \circ_j b)
  \end{array}
  \]
is commutative whenever \(\circ_j\) is defined.
\end{itemize}

\textbf{Proof.} By Prop. \((12.2.11)\) a map of colored cyclic operads is an entrywise map that preserves all the structure maps \(\gamma_T\) for \(T \in \text{Tree}(\mathcal{C})\). By the decomposed form of \(\gamma_T\) in \((12.4.4)\), it is enough for \(f\) to preserve the equivariant structure, the colored units, and the operations \(\circ_j\). \[\square\]
12.5. Applications: Cofibrant Resolutions of Cyclic Operads

Let us now provide some illustrations of the cofibrant resolution in Theorem 12.3.3(4) and the coherence of cyclic operads in Theorem 12.4.2.

Example 12.5.1 (Deligne-Grothendieck-Knudsen moduli spaces). Consider the cofibrantly generated monoidal model category $\mathcal{M} = \text{Top}$ of compactly generated topological spaces [Hov99] (Theorem 2.4.19) equipped with the commutative interval $J = [0, 1]$ with the the multiplication $\mu = \text{max}$. With $C$ being the one-point set, a space $X \in \text{Top}_{\Sigma^{op}}$ is cofibrant if and only if it is $\Sigma$-cofibrant and the unit map is a cofibration.

For $n \geq 0$, a stable $n$-pointed curve of genus $g$ is a tuple $(C, x_1, \ldots, x_n)$ in which:

- $C$ is a projective curve of arithmetic genus $g$, possibly with nodal singularities;
- the $x_i$’s, called marked points, are distinct smooth points in $C$ such that $C$ has no infinitesimal automorphisms preserving them.

Denote by $\overline{\mathcal{M}}_{g,n}$ the Deligne-Grothendieck-Knudsen moduli space of stable $n$-pointed curves of genus $g$ [Del72, Knu83]. Define the space $\mathcal{M}(n)/\{\ast\}$ if $n = 1$, $\{\ast\} \cup \bigsqcup_{g > 0} \mathcal{M}_{g,n}$ if $n > 2$.

Its $\Sigma_n$-action is given by permutation of marked points. The structure maps

$$\overline{\mathcal{M}}(m) \otimes \overline{\mathcal{M}}(n) \overset{\otimes_j}{\longrightarrow} \overline{\mathcal{M}}(m + n - 2)$$

with $m + n > 2$ are induced by gluing of curves along the $i$th marked point of one curve and the $j$th marked point of the other curve [GK94] (1.4.4). Then

$$\overline{\mathcal{M}} = \{\overline{\mathcal{M}}(n)\}_{n \geq 1}$$

is a one-colored topological cyclic operad [GK95]. Note that our indexing is shifted by 1 from [GK95]; our $\overline{\mathcal{M}}(n)$ is their $\mathcal{M}(n)$.

Since the unit $\{\ast\} \longrightarrow \overline{\mathcal{M}}(2)$ is the inclusion of the disjoint base point $\ast$, it is a cofibration. The $\Sigma$-action permutes the marked points, so $\overline{\mathcal{M}}$ is $\Sigma$-cofibrant. Therefore, Theorem 12.3.3(4) applies to yield a cofibrant resolution

$$W^{\Sigma}(\mathcal{M}) \overset{\eta}{\longrightarrow} \overline{\mathcal{M}}$$

of the one-colored topological cyclic operad $\overline{\mathcal{M}}$.

Example 12.5.2 (Riemann spheres with holes). Suppose $\mathcal{M} = \text{Top}$ and $J = ([0, 1], \text{max})$ as in Example 12.5.1. For $n > 2$ denote by $\overline{\mathcal{M}}_0(n)$ the moduli space of Riemann spheres with $n$ holes, i.e., of complex analytic embeddings

$$\bigsqcup_{k=1}^{n} D \overset{\phi}{\longrightarrow} S^2$$
in which $D$ is the closed unit disk in the complex plane. The images of the closed unit disks are called **parametrized holes**.

Define

$$\mathcal{M}_0(1) = \emptyset \quad \text{and} \quad \mathcal{M}_0(2) = \{\ast\}.$$ 

Then$$\mathcal{M}_0 = \{\mathcal{M}_0(n)\}_{n \geq 1}$$
is a one-colored topological cyclic operad \cite{GK95} whose equivariant structure is given by permutation of the parametrized holes. The structure maps

$$\mathcal{M}_0(m) \otimes \mathcal{M}_0(n) \overset{\varphi_{i,j}}{\longrightarrow} \mathcal{M}_0(m + n - 2)$$

with $m + n > 2$ are induced by sewing along the boundaries of the $i$th parametrized sphere (of the first Riemann sphere with $m$ holes) and the $j$th parametrized sphere (of the second Riemann sphere with $n$ holes). Our indexing is again shifted by 1 from \cite{GK95}; our $\mathcal{M}_0(n)$ is their $\mathcal{M}_0(n - 1)$.

The unit map $\{\ast\} \longrightarrow \mathcal{M}_0(2)$ is the identity map, hence a cofibration. The equivariant structure is given by permutation of the labels of the parametrized holes, so $\mathcal{M}_0$ is $\Sigma$-cofibrant. Therefore, Theorem 12.3.3(4) applies to yield a cofibrant resolution

$$W^{\text{cyc}}([0,1], \mathcal{M}_0) \overset{\eta}{\longrightarrow} \mathcal{M}_0$$
of the one-colored topological cyclic operad $\mathcal{M}_0$.

**Example 12.5.3 (Riemann spheres with colored holes).** Let $\mathbb{R}^+$ denote the set of positive real numbers. Then there is an $\mathbb{R}^+$-colored topological cyclic operad $\mathcal{M}_0^\mathbb{R}^+$ that is the $\mathbb{R}^+$-colored version of the one-colored topological cyclic operad $\mathcal{M}_0$ in Example 12.5.2. In $\mathcal{M}_0^\mathbb{R}^+$, each copy of the closed unit disk is replaced by any closed disk $D_r$ with radius $r > 0$. The sewing operation $\varphi_{i,j}$ is now only defined if the two relevant disks have the same radius. Once again Theorem 12.3.3(4) yields a cofibrant resolution

$$W^{\text{cyc}}([0,1], \mathcal{M}_0^\mathbb{R}^+) \overset{\eta}{\longrightarrow} \mathcal{M}_0^\mathbb{R}^+$$
of the $\mathbb{R}^+$-colored topological cyclic operad $\mathcal{M}_0^\mathbb{R}^+$.

**Example 12.5.4 (Chain cyclic operads).** Suppose $\text{Ch} = \text{Ch}(k)$ is the cofibrantly generated monoidal model category of chain complexes over a field $k$ of characteristic zero \cite{Hov99} (Theorem 2.3.11). It is equipped with the commutative interval $J = N\Delta^1$, where $\Delta^1$ is the commutative interval in simplicial sets given by the maximum operation on $\{0,1\}$, and $N$ is the normalized chain functor. Maschke’s Theorem implies that every object in $\text{Ch}(k)^{\Sigma^{op}}$ is cofibrant. Therefore, Theorem 12.3.3(4) applies to yield a cofibrant resolution of a colored cyclic operad as soon as the colored units are cofibrations, i.e., inclusions.

For examples, the chain associative operad and the chain $A_\infty$-operad can both be extended to one-colored cyclic operads \cite{GK95} (Prop. 2.4 and Prop. 2.8). The same is true for the chain commutative operad and the chain Lie operad \cite{GK95} (3.9). Since their unit maps are inclusions, Theorem 12.3.3(4) yields the cofibrant
resolutions

\[ W^{\text{cyc}}(N\Delta^1, \text{As}) \xrightarrow{\eta} \text{As} , \quad W^{\text{cyc}}(N\Delta^1, A_\infty) \xrightarrow{\eta} A_\infty , \]

\[ W^{\text{cyc}}(N\Delta^1, \text{Com}) \xrightarrow{\eta} \text{Com} , \quad W^{\text{cyc}}(N\Delta^1, \text{Lie}) \xrightarrow{\eta} \text{Lie} \]

of one-colored chain cyclic operads.
CHAPTER 13

Boardman-Vogt Resolutions of Colored Modular Operads

In this chapter we construct the Boardman-Vogt resolution of colored modular operads and observe that they have nice homotopical properties. One-colored modular operads were introduced in [GK98]. Surveys of one-colored modular operads are in [Mar08] Section 7 and [MSS02] Sections II.5.3–II.5.7. As in earlier sections, each entry of our Boardman-Vogt construction of a colored modular operad is defined in one step as a coend indexed by a substitution category, rather than as a sequential colimit of pushouts.

In Section 13.1 we describe the relevant graphs, called stable graphs, for colored modular operads. In Section 13.2 we define colored modular operads as algebras over a monad parametrized by stable graphs. In Section 13.3 we discuss the Boardman-Vogt resolutions of colored modular operads. In Section 13.4 we prove a coherence theorem for colored modular operads, characterizing them in terms of just a few generating operations and generating axioms. In Section 13.5 we discuss cofibrant resolutions of chain modular operads and of some well-known modular operads constructed from moduli spaces of stable curves with marked points and of Riemann spheres with holes.

13.1. Stable Graphs

Compared to unrooted trees (Def. 12.1.1), the stable graphs below are in some sense even simpler because we no longer need to think about exceptional edges. However, stable graphs are allowed to have an empty set of legs, which cannot happen in unrooted trees. There is also a bit more bookkeeping for stable graphs involving the genus functions. It is necessary to make sure that certain constructions, such as graph substitution, preserve the genus of a stable graph.

Recall that an ordinary connected graph is a connected graph (Def. 1.1.13) that is neither an exceptional edge (Example 1.1.6) nor an exceptional loop (Example 1.1.7), so it must have at least one vertex. For a graph $G$, its sets of internal edges and vertices are denoted $|G|$ and $Vt(G)$, respectively, while the set of edges is denoted $Ed(G)$. For an ordinary connected graph $G$, there is a decomposition

$$Ed(G) = |G| \cup \text{Leg}(G),$$

so an edge is either an internal edge or a leg. Recall that a vertex $v$ in a graph is by definition a finite set of flags. Its cardinality is denoted $|v|$. If $v$ is a vertex in a $\mathcal{C}$-colored ordinary connected graph with an ordering at $v$, then the ordering and the $\mathcal{C}$-coloring yield a $\mathcal{C}$-profile, which we will denote by $v$ as well. We write $\mathbb{N}$ for the set $\{0, 1, 2, \ldots \}$ of non-negative integers.
Convention 13.1.1. Throughout this section, every ordinary connected graph is equipped with a \( \mathcal{C} \)-coloring for a fixed non-empty set \( \mathcal{C} \) and a listing (Def. 1.2.9).

Definition 13.1.2. Suppose \( \mathcal{C} \) is a set.

1. A pair \( (p, \mathcal{C}) \in \mathbb{N} \times \text{Prof}(\mathcal{C}) \) is said to be stable if
   \[
   2p + |\mathcal{C}| - 2 > 0. 
   \]
   Denote by \( \text{St}(\mathbb{N}, \mathcal{C}) \) the set of stable pairs in \( \mathbb{N} \times \text{Prof}(\mathcal{C}) \).

2. A stable graph is a pair \( (G, g) \) consisting of
   - an ordinary connected graph \( G \), and
   - a genus function \( g : Vt(G) \rightarrow \mathbb{N} \) that satisfies the stability condition:
     \[
     (g(v), v) \in \text{St}(\mathbb{N}, \mathcal{C}) \quad \text{for} \quad v \in Vt(G). 
     \]

3. For an ordinary connected graph \( G \) equipped with a genus function \( g \), its genus is defined as
   \[
   g(G) = b_1(G) + \sum_{v \in Vt(G)} g(v), 
   \]
   where
   \[
   b_1(G) = |\mathcal{C}| - |Vt(G)| + 1. 
   \]

4. An isomorphism of stable graphs is an isomorphism of ordinary connected graphs that preserves the edge coloring function, the ordering (at each vertex and on the set of legs), and the genus function.

5. The set of isomorphism classes of stable graphs with genus \( p \) and profile \( \mathcal{C} \in \text{Prof}(\mathcal{C}) \) is denoted \( \text{StGr}(p, \mathcal{C}) \).

Remark 13.1.5. When \( \mathcal{C} \) is the one-element set, our stable graphs above differ from those in [GK98] 2.8 and [Mar08] Section 7 in that each of our graphs is equipped with an ordering of the finite set of flags in each vertex, while the graphs in [GK98] and [Mar08] do not have such an ordering. As in the case of cyclic operads, this difference means that, while modular operads in [GK98] and [Mar08] are defined on objects that already have an equivariant structure, our colored modular operads are defined on families of objects that do not have an equivariant structure. In our setting, the equivariant structure is a consequence of certain permuted corollas, which we will discuss below.

Remark 13.1.6. The definition of \( b_1(G) \) above is of course from Euler’s formula for finite connected planar graphs. If we were to define a geometric realization of an ordinary connected graph \( G \), then \( b_1(G) \) is the first Betti number. Geometrically, \( b_1(G) \) is the number of independent cycles in \( G \). Note, however, that our definition of \( b_1(G) \) does not rely on any notion of a geometric realization.

Example 13.1.7. For a pair \( (p, \mathcal{C}) \in \mathbb{N} \times \text{Prof}(\mathcal{C}) \), the stability condition (13.1.3) simply means:

1. If \( p = 0 \), then \( |\mathcal{C}| > 2 \).
2. If \( p = 1 \), then \( |\mathcal{C}| > 0 \).

In particular, if \( p \geq 2 \), then \( |\mathcal{C}| = 0 \) yields the stable pair \( (p, \emptyset) \).
Example 13.1.8. For each \((p, \mathcal{C}) \in \text{St}(\mathbb{N}, \mathcal{C})\) with \(\mathcal{C} = (c_1, \ldots, c_m)\), there is a \((p, \mathcal{C})\)-corolla

\[ C_{(p, \mathcal{C})} \in \text{StGr}(p, \mathcal{C}) \]

with one vertex \(v = \{l_1, \ldots, l_m\}\) consisting of \(m\) legs such that each flag \(l_i\) has color \(c_i\). For both the vertex \(v\) and \(C_{(p, \mathcal{C})}\) itself, the leg \(l_i\) is ordered by \(i\). The genus function is given by \(g(v) = p\), so the genus of the whole graph is

\[ g(C_{(p, \mathcal{C})}) = g(v) = p \]

because

\[ b_1(C_{(p, \mathcal{C})}) = 0. \]

(1) If \(\mathcal{C}\) is non-empty, then \(C_{(p, \mathcal{C})}\) is the \(\mathcal{C}\)-corolla in Example 12.1.3 with the additional structure \(g(v) = p\).

(2) If \(\mathcal{C}\) is empty, then \(p > 1\), and \(C_{(p, \emptyset)}\) is an isolated vertex \(o\), which is not an unrooted tree, with genus \(p\).

Example 13.1.9. With the same setting as in Example 13.1.8, for each permutation \(\sigma \in \Sigma_m\), there is a permuted corolla \(C_{(p, \mathcal{C})\sigma} \in \text{StGr}(p, \mathcal{C}\sigma)\), where \(\mathcal{C}\sigma = (c_{\sigma(1)}, \ldots, c_{\sigma(n)})\). Everything is the same as in the \((p, \mathcal{C})\)-corolla, except that the leg \(l_i\) is ordered by \(\sigma^{-1}(i)\). In other words, the \(i\)th leg in the permuted corolla \(C_{(p, \mathcal{C})\sigma}\) is \(l_{\sigma(i)}\). If \(\mathcal{C}\) is non-empty, then \(C_{(p, \mathcal{C})\sigma}\) is the permuted corolla \(C_{\mathcal{C}\sigma}\) in Example 12.1.4 with the additional structure \(g(v) = p\).

Recall the definition of \(\mathcal{a}_i \circ_j \mathcal{b}\) from Def. 12.1.6.

Example 13.1.10. Suppose \(a = (a_1, \ldots, a_m)\) and \(b = (b_1, \ldots, b_n)\) are \(\mathcal{C}\)-profiles, \(1 \leq i \leq m\), \(1 \leq j \leq n\), and \(a_i = b_j\). Suppose \((p, a), (q, b) \in \text{St}(\mathbb{N}, \mathcal{C})\). Then we may define the barbell

\[ T_{i,j}^{(p,a),(q,b)} \in \text{StGr}(p + q, a \circ_j b) \]

exactly as \(T_{i,j}^{a,b}\) in Example 12.1.10 with the genus function given by

\[ g(u) = p \quad \text{and} \quad g(v) = q. \]

Since

\[ b_1(T_{i,j}^{(p,a),(q,b)}) = 0, \]

the genus of the whole graph is

\[ g(T_{i,j}^{(p,a),(q,b)}) = g(u) + g(v) = p + q. \]

In the case \(m = n = i = j = 1\), \(p \geq 1\), and \(q \geq 1\), we have the barbell

\[ T_{1,1}^{(p,a),(q,a)} \in \text{StGr}(p + q, \emptyset), \]

whose underlying \(\mathcal{C}\)-colored connected graph looks like:

```
  u ------ a ------ v
```

This graph is not an unrooted tree because it does not have any legs.
Example 13.1.11. Suppose \((p, c) \in \text{St}(N, \mathcal{C})\) with \(c = (c_1, \ldots, c_m)\), \(m \geq 2\), and \(c_i = c_j\) for some \(1 \leq i \neq j \leq m\). There is a \textit{contracted corolla}

\[
\xi^{ij} C_{(p, c)} \in \text{StGr}(p + 1, \mathcal{C} \setminus \{c_i, c_j\})
\]

with one vertex \(v = \{l_1, \ldots, l_m\}\) and coloring \(\kappa(l_k) = c_k\). At the vertex \(v\), the flag \(l_k\) is ordered by \(k\). The flags \(l_i\) and \(l_j\) form an internal edge, which is therefore a loop at \(v\). All other flags are legs with leg ordering

\[
\chi(l_k) = \begin{cases} 
  k & \text{if } 1 \leq k < \min\{i, j\}, \\
  k - 1 & \text{if } \min\{i, j\} < k < \max\{i, j\}, \\
  k - 2 & \text{if } \max\{i, j\} < k \leq m.
\end{cases}
\]

The genus function is given by \(g(v) = p\), so the genus of the whole graph is

\[
g(\xi^{ij} C_{(p, c)}) = b_1(\xi^{ij} C_{(p, c)}) + g(v) = 1 + p.
\]

The underlying connected graph may be depicted as:

\[
\begin{array}{c}
  c_1 \\
  \ddots \\
  c_m
\end{array}
\quad
\begin{array}{c}
  \circ \\
  \circ \\
  \circ
\end{array}
\quad
\begin{array}{c}
  c_i \\
  \ddots \\
  c_j
\end{array}
\]

This graph is not simply-connected, so it is not an unrooted tree.

Colored modular operads will be defined as algebras over a monad whose multiplication corresponds to graph substitution of stable graphs.

Definition 13.1.12. Suppose given a stable graph \(G \in \text{StGr}(p, \mathcal{C})\) with genus function \(g_G\) and, for each vertex \(v \in G\), a stable graph \(H_v \in \text{StGr}(g_G(v), v)\). The \textit{graph substitution} \(G(H_v)\) is the stable graph whose underlying ordinary connected graph is the graph substitution in Def. 1.3.4. Its genus function is induced by those of the \(H_v\)'s.

By \(\text{YJ15}\) Prop. 6.12 ordinary connected graphs are closed under graph substitution. The stability of a graph is defined locally at each vertex. So if each \(H_v\) is stable, then so is \(G(H_v)\). We already know the graph substitution \(G(H_v)\) has the same profile as \(G\). We will show that the graph substitution \(G(H_v)\) has the same genus as \(G\) as well. To prove this we will use the following observation regarding generation of ordinary connected graphs with genus functions.

Proposition 13.1.13. Every ordinary connected graph with a genus function can be expressed as an iterated graph substitution involving

- \textit{permuted corollas} (Example 13.1.9),
- \textit{barbells} (Example 13.1.10), and
- \textit{contracted corollas} (Example 13.1.11).

Proof. Suppose \(G\) is an ordinary connected graph with a genus function \(g_G\). We proceed by induction on the number of internal edges \(|G|\). If \(|G| = 0\), then \(G\) has only one vertex and no loops, so it is a permuted corolla.

Suppose \(|G| \geq 1\). Pick an internal edge \(e\) in \(G\), which may be a loop at some vertex. We can decompose \(G\) as

\[
G = (G/e)(H)
\]
in which $G/e$ is obtained from $G$ by shrinking away the internal edge $e$, i.e., by removing the two flags constituting $e$.

(1) If $e$ is not a loop, then the two adjacent vertices, say $u$ and $v$ in $G$, minus the two flags in $e$, are combined into one vertex $w$, which is then given some ordering and genus

$$g_{G/e}(w) = g_G(u) + g_G(v).$$

The graph $H$ is the barbell $T$ defined by $u$, $v$, and $e$, except possibly for its leg ordering, and is substituted into $w$ in $G/e$. So $H$ has the form

$$H = (C\sigma)(T)$$

with $C\sigma$ the permuted corolla that corrects the leg ordering from that of $T$ to the ordering at $w$.

(2) If $e$ is a loop at a vertex $w$ in $G$, then the resulting vertex $w'$ in $G/e$ is given the induced ordering—i.e., the ordering at $w$ minus the two flags in $e$—and genus

$$g_{G/e}(w') = g(w) + 1.$$

The graph $H$ is the contracted corolla with unique vertex $w$ and one loop $e$ and is substituted into $w'$ in $G/e$.

A corolla is substituted into every vertex in $G/e$ not equal to $w$.

Since $G/e$ has one fewer internal edge than $G$, the induction hypothesis implies that it is generated by permuted corollas, barbells, and contracted corollas. The above decomposition of $G$ now proves the induction step. □

**Example 13.1.14.** Suppose $a, b, c, d \in \mathbb{C}$. Consider the stable graph $G$

$$
\begin{array}{c}
  c \\
  a \\
  \hline
  b \\
  d \\
\end{array}
$$

whose genus function is given by $g(u) = p \geq 0$. The genus of $G$ is $p + 2$. Then the graph substitutions $G_1(H_r)$ and $G_2(H_s)$, depicted as

$$
\begin{array}{c}
  H_r \\
  G_1 \\
  H_s \\
  G_2 \\
\end{array}
$$

are both equal to the stable graph $G$, where

$$g(r) = p + 1 = g(s).$$

We have

$$g(H_r) = p + 1 = g(H_s) \quad \text{and} \quad g(G_1) = p + 2 = g(G_2).$$

Note that $G_1$, $G_2$, $H_r$, and $H_s$ are all contracted corollas. The equality

$$G_1(H_r) = G_2(H_s)$$

tells us that two loops at a vertex can be created by substituting a contracted corolla into another contracted corolla. Moreover, the two loops can be created in either order.
Example 13.1.15. Suppose $b, c, d \in C$. Consider the stable graph $G$

whose genus function is given by

$$g(u) = p \quad \text{and} \quad g(v) = q$$

with $p, q \geq 0$. The genus of $G$ is $p + q + 1$. Then the graph substitutions $G_1(H_y)$ and $G_2(H_z)$, depicted as

are both equal to the stable graph $G$, where

$$g(y) = p + 1 \quad \text{and} \quad g(z) = p + q.$$ 

We have

$$g(H_y) = p + 1, \quad g(G_1) = p + q + 1 = g(G_2), \quad \text{and} \quad g(H_z) = p + q.$$ 

Observe that both $G_1$ and $H_z$ are barbells, and both $H_y$ and $G_2$ are contracted corollas. The equality

$$G_1(H_y) = G_2(H_z)$$

tells us that a contracted corolla substituted into a barbell can be rewritten as a barbell substituted into a contracted corolla.

Example 13.1.16. Suppose $a, b, c, d, e, f \in C$. Consider the stable graph $G$

whose genus function is given by

$$g(u) = p \quad \text{and} \quad g(v) = q$$

with $p, q \geq 0$. The genus of $G$ is $p + q + 1$. Then the graph substitutions $G_1(H_w)$ and $G_2(H_x)$, depicted as
are both equal to the stable graph $G$, where

$$g(w) = p + q = g(x).$$

We have

$$g(H_w) = p + q = g(H_x) \quad \text{and} \quad g(G_1) = p + q + 1 = g(G_2).$$

Observe that both $H_w$ and $H_x$ are barbells, and both $G_1$ and $G_2$ are contracted corollas. The equality

$$G_1(H_w) = G_2(H_x)$$

tells us that two internal edges connecting the same pair of distinct vertices can be created by substituting a barbell into a contracted corolla. Furthermore, we can create the two internal edges in either order.

In Examples [13.1.14], [13.1.15], and [13.1.16] observe that each of $G_1$ and $G_2$ has the same genus as $G$. This suggests that the genus of a stable graph does not change when another stable graph is substituted into one of its vertices. We will prove this below.

**Proposition 13.1.17.** In the context of Def. [13.1.12] the graph substitution $G(H_v)$ has the same genus as $G$.

**Proof.** By the unity and associativity of graph substitution and Prop. [13.1.13] we may assume that:

1. $G$ is among the three types of generating graphs: permuted corollas, barbells, and contracted corollas.
2. For a chosen vertex $v$ in $G$, $H_v$ is among the three types of generating graphs, and $H_u$ is a corolla for each vertex $u \neq v$ in $G$.

So we only need to check nine cases. Recall that a permuted corolla has the same genus as its unique vertex because $b_1 = 0$. Graph substitution involving a permuted corolla only changes the order at a vertex (if $H_v$ is a permuted corolla) or the order of the legs (if $G$ is a permuted corolla), and neither operation changes the value of $b_1$. It follows that the assertion is true if either $G$ or $H_v$ is a permuted corolla.

The genus of a barbell is the sum of the genera of its two vertices because $b_1 = 0$. Likewise, the genus of a graph substitution involving two barbells is the sum of the genera of its three vertices because $b_1 = 0$. So the assertion is true if both $G$ and $H_v$ are barbells.

If one of them is a barbell and the other one is a contracted corolla, then the assertion follows from the argument in Examples [13.1.15] and [13.1.16]. If both $G$ and $H_v$ are contracted corollas, then the assertion follows from the argument in Example [13.1.14].
13.2. Colored Modular Operads

For a given set \( \mathcal{C} \), we now define \( \mathcal{C} \)-colored modular operads as algebras over a monad indexed by stable graphs. As before \((M, \otimes, \mathbb{I})\) is a cocomplete symmetric monoidal category whose monoidal product commutes with colimits on both sides. Recall from Def. 13.1.2 that \( \text{St}(\mathbb{N}, \mathcal{C}) \) is the subset of \( \mathbb{N} \times \text{Prof}(\mathcal{C}) \) consisting of stable pairs and that \( \text{StGr}(p, \mathcal{C}) \) is the set of isomorphism classes of stable graphs with genus \( p \) and profile \( \mathcal{C} \).

**Definition 13.2.1.** For \( X \in M^{\text{St}(\mathbb{N}, \mathcal{C})} \) and \( G \in \text{StGr}(p, \mathcal{C}) \) with genus function \( g_G \), define the object
\[
X[G] = \bigotimes_{v \in G} X(g_G(v), v)
\]
in which the unordered tensor product is indexed by the set of vertices in \( G \).

**Definition 13.2.2.** Define a monad \( (F^{\text{mod}}, \mu, \nu) \) on \( M^{\text{St}(\mathbb{N}, \mathcal{C})} \) as follows.

**The functor:** Define the functor
\[
F^{\text{mod}} : M^{\text{St}(\mathbb{N}, \mathcal{C})} \rightarrow M^{\text{St}(\mathbb{N}, \mathcal{C})}
\]
by
\[
F^{\text{mod}}(p, \mathcal{C}) = \bigsqcup_{G \in \text{StGr}(p, \mathcal{C})} \mathcal{P}[G]
\]
for \( p \in M^{\text{St}(\mathbb{N}, \mathcal{C})} \) and \( (p, \mathcal{C}) \in \text{St}(\mathbb{N}, \mathcal{C}) \).

**The multiplication:** For each stable graph \((G, g_G)\), we have
\[
F^{\text{mod}}[G] = \bigotimes_{v \in G} F^{\text{mod}}(g_G(v), v)
\]
\[
= \bigotimes_{v \in G} \bigsqcup_{H_v \in \text{StGr}(g_G(v), v)} \mathcal{P}[H_v]
\]
\[
= \bigsqcup_{\{H_v\} \subseteq G} \bigotimes_{v \in G} \mathcal{P}[H_v].
\]

The monadic multiplication \( \mu_p \) is defined by the commutative diagrams
\[
\begin{array}{ccc}
\bigsqcup_{G, (H_v) \in G} \bigotimes_{v \in G} \mathcal{P}[H_v] & \xrightarrow{\text{inclusion}} & \bigotimes_{v \in G} \mathcal{P}[H_v] \\
\downarrow z & & \downarrow z \\
F^{\text{mod}} F^{\text{mod}} \mathcal{P}(p, \mathcal{C}) = \bigsqcup_{G \in \text{StGr}(p, \mathcal{C})} F^{\text{mod}}[G] & \xrightarrow{\text{inclusion}} & \mathcal{P}[G(H_v)] \\
\mu_p & & \downarrow \text{inclusion} \\
F^{\text{mod}} \mathcal{P}(p, \mathcal{C}) = \bigsqcup_{K \in \text{StGr}(p, \mathcal{C})} \mathcal{P}[K] & \xrightarrow{\text{inclusion}} & F^{\text{mod}} \mathcal{P}(p, \mathcal{C})
\end{array}
\]

for \( (p, \mathcal{C}) \in \text{St}(\mathbb{N}, \mathcal{C}) \), \((G, g_G) \in \text{StGr}(p, \mathcal{C})\), and \( H_v \in \text{StGr}(g_G(v), v) \) with \( v \in G \). The graph substitution \( G(H_v) \) and \( G \) have the same profiles and the same genus by Prop. 13.1.17, so the bottom inclusion is well-defined.

**The unit:** The monadic unit is defined by the corolla inclusion
\[
\mathcal{P}(p, \mathcal{C}) = \mathcal{P}[C(p, \mathcal{C})] \xrightarrow{\text{inclusion}} F^{\text{mod}} \mathcal{P}(p, \mathcal{C})
\]
13.2. COLORED MODULAR OPERADS

in which \( C_{(p,c)} \) is the \((p,c)\)-corolla in Example 13.1.8.

The proof that \((F^{\text{mod}}, \mu, \nu)\) is actually a monad is exactly as in \[\text{YJ15}\] Theorem 10.38, which was written for a pasting scheme. Associativity and unity of the monad are consequences of those of graph substitution of stable graphs.

Example 13.2.3. Suppose \( P \in M^{\text{St}(\mathbb{N}, \mathcal{C})} \).

1. For a permuted corolla as in Example 13.1.9 we have

\[ P[C_{(p,c)}] = P(p,c). \]

2. For a barbell as in Example 13.1.10 we have

\[ P[T_{i,j}^{(p,a),(q,b)}] = P(p,a) \odot P(q,b), \]

where \( \odot \) means unordered tensor product.

3. For a contracted corolla as in Example 13.1.11 we have

\[ P[\xi_{ij} C_{(p,c)}] = P(p,c). \]

Definition 13.2.4. Given a set \( \mathcal{C} \), the category of algebras over the monad \((F^{\text{mod}}, \mu, \nu)\) on \( M^{\text{St}(\mathbb{N}, \mathcal{C})} \) is denoted \( M_{\text{mod}}^{\mathcal{C}} \). Its objects are called \( \mathcal{C}\)-colored modular operads in \( M \).

Remark 13.2.5. As in the cyclic case, when \( \mathcal{C} \) is the one-point set, a \( \{\ast\}\)-colored modular operad is exactly a modular operad in \[\text{GK98}\], in which modular operads are defined as algebras over a monad \( M \) indexed by some stable graphs. However, in \[\text{GK98}\] their stable graphs do not have an ordering on the set of flags in each vertex. So their monad \( M \) is defined on a category whose objects already have an equivariant structure. In contrast, our stable graphs come equipped with an ordering on the set of flags in each vertex. So our monad \( F^{\text{mod}} \) is defined on the category \( M^{\text{St}(\mathbb{N}, \mathcal{C})} \) in which an object is just a \( \text{St}(\mathbb{N}, \mathcal{C})\)-indexed family of objects in \( M \) without any equivariant structure. In our setting, the equivariant structure comes from the permuted corollas in Example 13.1.9.

Unraveling the definition of the monad \( F^{\text{mod}} \), we have the following more explicit description of a colored modular operad. The one-colored case of the following observation is \[\text{GK98}\] Prop. 2.23.

Proposition 13.2.6. A \( \mathcal{C}\)-colored modular operad is exactly a pair \((P, \gamma^P)\) consisting of:

- an object \( P \in M^{\text{St}(\mathbb{N}, \mathcal{C})} \);
- a structure map

\[ P[G] \xrightarrow{\gamma^P_{(p,c)}} P(p,c) \]

for each \((p,c) \in \text{St}(\mathbb{N}, \mathcal{C})\) and each \( G \in \text{StGr}(p,c) \).

This data is subject to the following two conditions.

Unity: For each \((p,c) \in \text{St}(\mathbb{N}, \mathcal{C})\), the structure map \( \gamma^P_{C_{(p,c)}} \) is the identity map of \( P(p,c) \), where \( C_{(p,c)} \) is the \((p,c)\)-corolla in Example 13.1.8.
Associativity: The diagram

\[
\begin{align*}
\otimes_{v \in G} P[H_v] \xrightarrow{\otimes_{v \in G} \gamma^P_{H_v}} \otimes_{v \in G} P(g_G(v), v) &= P[G] \\
\downarrow & \\
P[G(H_v)] \xrightarrow{\gamma^P_G} P(p, \xi) & \cong \\
\end{align*}
\]

is commutative for all \((p, \xi) \in \text{St}(\mathbb{N}, \mathcal{C}), (G, g_G) \in \text{StGr}(p, \xi),\) and \(H_v \in \text{StGr}(g_G(v), v)\) with \(v \in G.\)

Proof. For an \(F^\text{mod}\)-algebra \(P\) with structure map \(\gamma : F^\text{mod}P \longrightarrow P,\) the structure map \(\gamma^P_G\) is the composite

\[
\gamma^P_G
\]

for \(G \in \text{StGr}(p, \xi).\) The above unity and associativity conditions then correspond to those of an \(F^\text{mod}\)-algebra. \(\square\)

Example 13.2.8. Suppose \(P\) is a \(\mathcal{C}\)-colored modular operad.

1. For \((p, \xi) \in \text{St}(\mathbb{N}, \mathcal{C})\) and a permutation \(\sigma \in \Sigma_{[\xi]}\), there is a structure map

\[
P[C(p, \xi)\sigma] = P(p, \xi) \xrightarrow{\gamma^P_{p, \xi} \sigma} P(p, \xi) \sigma
\]

corresponding to the permuted corolla \(C(p, \xi)\sigma\) in Example 13.1.9. These structure maps yield the equivariant structure on \(P.\)

2. For \((p, a), (q, b) \in \text{St}(\mathbb{N}, \mathcal{C}), 1 \leq i \leq |a|, 1 \leq j \leq |b|,\) and \(a_i = b_j,\) there is a structure map

\[
P[T_{i,j}^p(a, b)] = P(p, a) \otimes P(q, b) \xrightarrow{\gamma^P_{p} \otimes \gamma^P_{q}} P(p + q, a_i \circ b_j)
\]

corresponding to the barbell in Example 13.1.10.

3. For \((p, \xi) \in \text{St}(\mathbb{N}, \mathcal{C})\) with \(1 \leq i \neq j \leq |\xi|\) and \(c_i = c_j,\) there is a structure map

\[
P[\xi_{ij} C(p, \xi)] = P(p, \xi) \xrightarrow{\gamma^P_{\xi_{ij} C(p, \xi)}} P(p + 1, \xi \setminus \{c_i, c_j\})
\]

corresponding to the contracted corolla in Example 13.1.11.

Next is the map version of Prop. 13.2.6.

Proposition 13.2.9. A map of \(\mathcal{C}\)-colored modular operads

\[
f : (P, \gamma^P) \longrightarrow (Q, \gamma^Q)
\]

is exactly a map

\[
f : P \longrightarrow Q \in M^{\text{St}(\mathbb{N}, \mathcal{C})}
\]
such that the diagram

\[
\begin{array}{ccc}
P[G] & \xrightarrow{\otimes} & Q[G] \\
\gamma_G & & \gamma_G \\
P(p,c) & \xrightarrow{f} & Q(p,c)
\end{array}
\]

is commutative for all \((p,c) \in \text{St}(\mathbb{N}, \mathcal{C})\) and \(G \in \text{StGr}(p, \mathcal{C})\).

Recall that \(\Sigma^\mathcal{C}\) is the groupoid of all \(\mathcal{C}\)-profiles with right permutations \(\sigma : x \mapsto x^{\sigma}\) as isomorphisms. The next concept is the modular operad analogue of a \(G_0\)-prop (Def. 4.2.1).

**Definition 13.2.10.**

1. Denote by \(\text{St}(\mathbb{N} \times \Sigma^\mathcal{C})\) the full sub-category of \(\mathbb{N} \times \Sigma^\mathcal{C}\) consisting of objects \((p,c) \in \text{St}(\mathbb{N}, \mathcal{C})\) and \(G \in \text{StGr}(p, \mathcal{C})\).

2. Denote by \(M_{\text{St}(\mathbb{N} \times \Sigma^\mathcal{C})} \xrightarrow{\text{F}_{\text{mod}}} M_{\text{mod}} \xleftarrow{U} \) the free-forgetful adjunction. The right adjoint \(U\) preserves all the entries and the structure maps corresponding to permuted corollas as in Example 13.2.8(1).

To describe the left adjoint \(F_{\text{mod}}\), we use the following indexing category.

**Definition 13.2.12.** For \((p,c) \in \text{St}(\mathbb{N}, \mathcal{C})\) define the extension category \(E_{\text{mod}}(p, \mathcal{C})\) as follows.

- Its object set is \(\text{StGr}(p, \mathcal{C})\).
- A map has the form \((H_v) : G(H_v) \rightarrow G\) in which each \(H_v \in \text{StGr}(g_G(v), v)\) is a permuted corolla as in Example 13.1.9.
- The identity of an object \(G \in \text{StGr}(p, \mathcal{C})\) is \((C_{(g_G(v), v)})_{v \in G}\) with each \(C_{(g_G(v), v)}\) the \((g_G(v), v)\)-corolla.
- Composition is defined by graph substitution.

Next is the modular operad analogue of Lemma 4.2.9. The proof is essentially identical to the pasting scheme case \([\text{YJ15}]\) (Lemmas 12.6 and 12.8).

**Lemma 13.2.13.** The left adjoint \(F_{\Sigma_{\text{mod}}} : M_{\text{St}(\mathbb{N} \times \Sigma^\mathcal{C})} \rightarrow M_{\text{mod}}\) is given entrywise by

\[
F_{\Sigma_{\text{mod}}} X(p, \mathcal{C}) = \colim_{G \in M_{\text{mod}}(p, \mathcal{C})} X[G]
\]

for \((p, \mathcal{C}) \in \text{St}(\mathbb{N}, \mathcal{C})\) and \(X \in M_{\text{St}(\mathbb{N} \times \Sigma^\mathcal{C})}\).

**Remark 13.2.14.** When \(\mathcal{C}\) is the one-element set, objects in \(M_{\text{St}(\mathbb{N} \times \Sigma^\mathcal{C})}\) are called stable \(S\)-modules in \([\text{GK98}]\) (2.1). So the left adjoint \(F_{\Sigma_{\text{mod}}}\) is isomorphic to the functor \(M\) in \([\text{GK98}]\) (2.20). However, our description of \(F_{\Sigma_{\text{mod}}}\) is a little bit different from \(M\) because our graphs are equipped with an ordering at each vertex, while those in \([\text{GK98}]\) are not.
13.3. Boardman-Vogt Resolution

For a fixed set \( \mathcal{C} \), we now define the Boardman-Vogt resolution of \( \mathcal{C} \)-colored modular operads in \((\mathcal{M}, \otimes, 1)\). As in the cyclic case, the definitions and proofs are essentially identical to our work in earlier sections, replacing a pasting scheme \( \mathcal{G} \) and pairs of \( \mathcal{C} \)-profiles with stable graphs and stable pairs in \( \text{St}(\mathbb{N}, \mathcal{C}) \). Therefore, we will only state some of the key definitions and results and omit the detailed proofs.

The \( W \)-construction for colored modular operads and some of the proofs are actually easier than in earlier sections. For example, for generalized props and colored cyclic operads, we needed a commutative segment \( J \) for the ordinary internal edges. The multiplication \( \mu : J \otimes 2 \to J \) was needed when an exceptional edge was substituted into a vertex connecting two internal edges. Due to the absence of exceptional edges among stable graphs, we no longer need the multiplication.

**Definition 13.3.1.** A **cylinder** in \( \mathcal{M} \) is a tuple \((J, 0, 1, \epsilon)\) consisting of an object \( J \) and maps

\[
\mathbb{1} \cup \mathbb{1} \xrightarrow{(0,1)} J \xrightarrow{\epsilon} \mathbb{1}
\]

such that

\[ \epsilon 0 = \text{Id}_1 = \epsilon 1. \]

If \( \mathcal{M} \) is a model category, then a **cofibrant cylinder** is a cylinder with \((0,1)\) a cofibration and \( \epsilon \) a weak equivalence.

Fix a cylinder \((J, 0, 1, \epsilon)\) in \( \mathcal{M} \).

**Definition 13.3.2.** Suppose \((p, c) \in \text{St}(\mathbb{N}, \mathcal{C})\).

1. Define the **substitution category** \( \text{StGr}(p, c) \) with object set \( \text{StGr}(p, c) \). A map

\[
(H_v) : G(H_v) \to G
\]

in \( \text{StGr}(p, c) \) is a family of stable graphs \( H_v \in \text{StGr}(g_G(v), v) \) for \( v \in G \).

This is well-defined by Prop. 13.1.17. The identity map of \( G \) is

\[
(C_{(g_G(v), v)}) : G = G(C_{(g_G(v), v)}) \to G
\]

with \( C_{(g_G(v), v)} \) the \((g_G(v), v)\)-corolla in Example 13.1.8. Composition is defined by graph substitution of stable graphs.

2. Define a functor \( J : \text{StGr}(p, c)^\text{op} \to \mathcal{M} \) by setting

\[
J[G] = \bigotimes_{e \in G} J = J^{\otimes |G|}.
\]

For a map \((H_v) : G(H_v) \to G \in \text{StGr}(p, c)\), the required map

\[
J[G] \to J[G(H_v)]
\]

is induced by the map \( 0 : \mathbb{1} \to J \) for each internal edge in each \( H_v \) (which must become an internal edge in \( G(H_v) \)).

**Definition 13.3.3.** Suppose \( P \) is a \( \mathcal{C} \)-colored modular operad in \( \mathcal{M} \), and \((p, c) \in \text{St}(\mathbb{N}, \mathcal{C})\).
(1) Define a functor \( P : \text{StGr}(p, \mathfrak{C}) \to M \) by setting
\[
P[G] = \bigotimes_{v \in G} P(g_G(v), v)
\]
for \((G, g_G) \in \text{StGr}(p, \mathfrak{C})\). For a map \((H_v) : G(H_v) \to G\) in \(\text{StGr}(p, \mathfrak{C})\), the map
\[
P[G(H_v)] = \bigotimes_{v \in G} P[H_v] \xrightarrow{\bigotimes \gamma^{P}_{H_v}} \bigotimes_{v \in G} P(g_G(v), v) = P[G]
\]
is the tensor product of the structure maps
\[
\gamma^{P}_{H_v} : P[H_v] \to P(g_G(v), v).
\]

(2) Define the coend
\[
W^{\text{mod}}(J, P)(p, \mathfrak{C}) = \int_{\text{Gr}(p, \mathfrak{C})}^G J[G] \otimes P[G].
\]
The family of objects \(W^{\text{mod}}(J, P)(p, \mathfrak{C})\) as \((p, \mathfrak{C})\) runs through \(\text{St}(\mathbb{N}, \mathfrak{C})\) is denoted \(W^{\text{mod}}(J, P)\).

(3) For \(G \in \text{StGr}(p, \mathfrak{C})\), define the map \(\gamma^{W^{\text{mod}}(J, P)}_{G}\) by the commutative diagrams
\[
\bigotimes_{v \in G} J[H_v] \otimes P[H_v] \xrightarrow{\pi} J[G(H_v)] \otimes P[G(H_v)]
\]
\[
\bigotimes_{v \in G} W^{\text{mod}}(J, P)(g_G(v), v)
\]
\[
\bigotimes_{v \in G} W^{\text{mod}}(J, P)[G] \xrightarrow{\gamma^{W^{\text{mod}}(J, P)}_{G}} W^{\text{mod}}(J, P)(p, \mathfrak{C})
\]
as \(\{H_v\}\) runs through \(\prod_{v \in G} \text{StGr}(g_G(v), v)\), where each \(\omega\) is the natural map. The map \(\pi\) is the natural isomorphism on the \(P\)-component and is \(\bigotimes_{E} 1\) on the \(J\)-component with \(E = |G(H_v)| \setminus \prod_{v \in G} |H_v|\).

**Theorem 13.3.4.** Suppose \(P\) is a \(\mathfrak{C}\)-colored modular operad in \(M\).

(1) When equipped with the structure maps \(\gamma^{W^{\text{mod}}(J, P)}\), \(W^{\text{mod}}(J, P)\) is a \(\mathfrak{C}\)-colored modular operad.

(2) There is a natural augmentation \(\eta : W^{\text{mod}}(J, P) \to P \in M^{\text{mod}}\) that is entrywise defined by the commutative diagrams
\[
J[G] \otimes P[G] \xrightarrow{\delta^e} \mathbb{I} \otimes [G] \otimes P[G] \cong P[G]
\]
\[
W^{\text{mod}}(J, P)(p, \mathfrak{C}) \xrightarrow{\eta} P(p, \mathfrak{C})
\]
for \((p, \mathfrak{C}) \in \text{St}(\mathbb{N}, \mathfrak{C})\) and \(G \in \text{StGr}(p, \mathfrak{C})\).
(3) The counit of \( P \) naturally factors as

\[
\begin{array}{ccc}
F_{\Sigma}^\text{mod} U P & \xrightarrow{\delta} & W^\text{mod}(J, P) \\
\downarrow & & \downarrow \\
\eta \in \mathcal{M}^\text{mod}
\end{array}
\]

with \( \delta \) uniquely determined by the commutative diagrams

\[
\begin{array}{ccc}
\downarrow & & \downarrow \\
F_{\Sigma}^\text{mod} U P(p, \xi) & \xrightarrow{\delta} & W^\text{mod}(J, P)(p, \xi)
\end{array}
\]

for \((p, \xi) \in \text{St}(\mathbb{N}, \mathcal{C})\) and \( G \in \text{StGr}(p, \xi)\).

(4) Suppose \( \mathcal{M} \) is a cofibrantly generated monoidal model category with a cofibrant cylinder \( J \) and a cofibrant \( 1 \) such that \( \mathcal{M}^\text{mod} \) inherits a model structure with entrywise weak equivalences and fibrations. Suppose \( P \in \mathcal{M}^\text{mod} \) such that \( U P \in \mathcal{M}^\text{St}(\mathbb{N}, \mathcal{C}^\text{op}) \) is cofibrant. Then the augmentation

\[
\eta : W^\text{mod}(J, P) \longrightarrow P
\]

is a cofibrant resolution of \( P \).

**Proof.** As in the cyclic case, we simply reuse the proofs of Theorem 3.5.17, Prop. 4.1.2, Theorem 4.2.14, and Theorem 7.3.2. For the stable graph analogue of the compatibility condition (Def. 7.2.5), observe that the stable graph analogue of the map \( \beta_G \) (resp., \( \beta_G \otimes \)) is an acyclic cofibration (resp., cofibration) in \( \mathcal{M} \) by a simple induction involving the pushout product axiom and the acyclic cofibration \( 0 : 1 \longrightarrow J \) (resp., cofibration \( (0, 1) : 1 \cup 1 \longrightarrow 1 \)). The reason is that in a stable graph, every internal edge can be shrunk.

Furthermore, since there are no exceptional edges among stable graphs, we no longer need to consider tunnels (Def. 5.2.2). In the current setting, the map \( \alpha_G \) in (5.2.3) is the identity map of \( P[G] \). The maps \( \delta_G \) and \( \delta_G \) in (5.2.10) are now the tensor products

\[
\begin{align*}
\delta_G = \beta_G \otimes P[G] : J^* G \otimes P[G] & \longrightarrow J[G] \otimes P[G], \\
\delta_G = \beta_G \otimes P[G] : J^* G \otimes P[G] & \longrightarrow J[G] \otimes P[G],
\end{align*}
\]

Recall the free-forgetful adjunction \( F_{\Sigma}^\text{mod} \to U \) in (13.2.11). The following observation is the modular analogue of Theorem 9.5.5.

**Theorem 13.3.5.** Suppose \( P \) is a \( \mathcal{C} \)-colored modular operad in \( \mathcal{M} \). Then there is a natural isomorphism

\[
(F_{\Sigma}^\text{mod} U)^{\bullet+1} P \stackrel{\simeq}{\longrightarrow} W^\text{mod}(\Delta^1_M, P)
\]

of simplicial \( \mathcal{C} \)-colored modular operads in \( \mathcal{M} \) augmented over \( P \).
13.4. Coherence of Colored Modular Operads

Here we describe colored modular operads in terms of a small number of generating operations and axioms. This is not needed for the Boardman-Vogt resolution of colored modular operads. This material is included for completeness and future reference. The first three generating axioms below are the modular analogues of the commutativity, equivariance, and associativity of \( \circ \) in the coherence of colored cyclic operads in Theorem 12.4.2.

**Theorem 13.4.1.** A \( \mathcal{C} \)-colored modular operad is exactly a tuple \((P, \circ, \xi)\) with:

- an equivariant object \( P \in \mathcal{M}^{\text{St}(\mathbb{N} \times \Sigma^n_\mathcal{C})} \);
- a structure map
  \[
P(p, a) \otimes P(q, b) \xrightarrow{\circ} P(p + q, a \circ_j b)
  \]
  with an ordered tensor product on the left, whenever \((p, a), (q, b) \in \text{St}(\mathbb{N}, \mathcal{C})\), \(1 \leq i \leq |a|, 1 \leq j \leq |b|\), and \(a_i = b_j\), with \(a_i \circ_j b\) as in \([12.1.7]\).
- a structure map
  \[
P(p, c) \xrightarrow{\xi^j} P(p + 1, c \setminus \{c_i, c_j\})
  \]
  whenever \((p, c) \in \text{St}(\mathbb{N}, \mathcal{C})\) with \(1 \leq i \neq j \leq |c|\) and \(c_i = c_j\).

This data is required to satisfy the following seven generating axioms.

1. **Commutativity of \( \circ \):** The diagram
   \[
   \begin{array}{ccc}
P(p, a) \otimes P(q, b) & \xrightarrow{\circ} & P(p + q, a \circ_j b) \\
   \xrightarrow{\text{switch}} & & \xrightarrow{\sigma} \\
P(q, b) \otimes P(p, a) & \xrightarrow{\circ} & P(p + q, b \circ_j a)
   \end{array}
   \]
   is commutative whenever it is defined. The right vertical map is the equivariant structure map for the block permutation
   \[
   \sigma = ((1 \, 3)(2 \, 4))(j - 1, |a| - i, i - 1, |b| - j) \in \Sigma_{|a|+|b|-2}
   \]
   induced by \((1 \, 3)(2 \, 4) \in \Sigma_4\) that permutes the four consecutive blocks of the indicated lengths. This is the permutation that satisfies
   \[
   (a_i \circ_j b)\sigma = b_j \circ_i a.
   \]

2. **Equivariance of \( \circ \):** The diagram
   \[
   \begin{array}{ccc}
P(p, a) \otimes P(q, b) & \xrightarrow{(\circ, \text{id})} & P(p, a\sigma) \otimes P(q, b) \\
   \xrightarrow{\circ_j} & & \xrightarrow{\sigma^{-1}(i) \circ_j} \\
P(p + q, a_i b) & \xrightarrow{\sigma'} & P(p + q, a \sigma \circ_j b)
   \end{array}
   \]
   is commutative whenever it is defined. The bottom horizontal map is the equivariant structure map for the block permutation
   \[
   \sigma' = \sigma(1, \ldots, 1, |b| - 1, 1, \ldots, 1) \in \Sigma_{|a|+|b|-2}
   \]
   with:
   \[
   (a_i \circ_j b)\sigma' = b_j \circ_i a.
   \]
induced by $\sigma \in \Sigma_{|q|}$ that regards the interval $[i, i + |b| - 2]$ as a single block. This is the permutation that satisfies

$$(\alpha_i \circ \beta_j) \sigma' = a \sigma \sigma^{-1}(i) \circ \beta_j b.$$ 

(3) **Associativity of $\circ$:** The diagram

$$\begin{align*}
\xymatrix{ & P(p, a) \otimes P(q, b) \otimes P(r, c) 
\ar[rr]^-{(\text{id}, \phi_i)} \ar[d]_-{(\phi_j, \text{id})} & & P(p, a) \otimes P(q + r, b \circ_i c) 
\ar[d]^-{\phi_t} \ar[rr]^-{\phi_j} & & P(p + q + r, d) 
\ar[rr]^-{\phi_j} & & P(p + q + r, d) }
\end{align*}$$

is commutative whenever it is defined with $1 \leq j \neq k \leq |b|$, in which

$$s = \begin{cases} |b| - j + i - 1 + k & \text{if } k < j, \\
|b| - j + i - 1 + k & \text{if } k > j, \\
i - 1 - j + k & \text{if } j > i,
\end{cases}$$

and

$$d = (\alpha_i \circ \beta_j) s \circ_i c = a \circ_t (b \circ_i c).$$

(4) **Commutativity of $\xi$:** The diagram

$$\begin{align*}
\xymatrix{ & P(p, a) 
\ar[r]^-{\xi_{ij}} & P(p + 1, a \setminus \{a_i, a_j\}) 
\ar[d]_-{\xi^{kl}} \ar[r]^-{\xi_{ij}'} & P(p + 2, a \setminus \{a_i, a_j, a_k, a_l\}) 
\ar[d]^-{\xi^{kl}}}
\end{align*}$$

is commutative whenever $(p, a) \in \text{St}(\mathbb{N}, \mathbb{C})$ with $i, j, k, l$ distinct in $\{1, \ldots, |a|\}$, $a_i = a_j$, and $a_k = a_l$. The indices $k', l' \in \{1, \ldots, |a| - 2\}$ correspond to the entries $a_{i'}, a_{j'}$ in $a \setminus \{a_i, a_j\}$. Similarly, the indices $i', j' \in \{1, \ldots, |a| - 2\}$ correspond to the entries $a_{i'}, a_{j'}$ in $a \setminus \{a_k, a_l\}$.

(5) **Equivariance of $\xi$:** The diagram

$$\begin{align*}
\xymatrix{ & P(p, a) 
\ar[r]^-{\xi_{ij}} & P(p + 1, a \setminus \{a_i, a_j\}) 
\ar[d]_-{\sigma} \ar[r]^-{\xi^{-1}_{ij} \circ \sigma^{-1}_{ij}} & P(p + 1, a \setminus \{a_i, a_j\}) 
\ar[d]^-{\sigma'} \ar[r]^-{\xi_{ij}'} & P(p + 1, a \setminus \{a_i, a_j\}) 
\ar[d]_-{\sigma} & & P(p + 1, a \setminus \{a_i, a_j\}) }
\end{align*}$$

is commutative whenever it is defined. The permutation $\sigma' \in \Sigma_{|a| - 2}$ is induced by $\sigma \in \Sigma_{|a|}$ by disregarding $a_i$ and $a_j$. This is the permutation that satisfies

$$a \sigma \setminus \{a_i, a_j\} = (a \setminus \{a_i, a_j\}) \sigma'.$$

(6) **$\xi$ commuting with $\circ$:** The diagram

$$\begin{align*}
\xymatrix{ & P(p, a) \otimes P(q, b) 
\ar[r]^-{(\xi_{kl}, \text{id})} \ar[d]_-{\phi_j} & P(p + 1, a \setminus \{a_k, a_l\}) \otimes P(q, b) 
\ar[d]^-{\phi_j} \ar[r]^-{\phi_{ij}'} & P(p + q + 1, d) 
\ar[r]^-{\phi_{ij}'} & P(p + q + 1, d) }
\end{align*}$$
is commutative whenever it is defined with \(i, k, l\) distinct in \(\{1, \ldots, |a|\}\) and
\[
d' = (a \setminus \{a_k, a_l\}) \circ \gamma \circ j = (a \setminus \{a_k, b_l\}) \setminus \{a_k, a_l\}.
\]
The index \(i' \in \{1, \ldots, |a| - 2\}\) corresponds to \(a_i\) in \(a \setminus \{a_k, a_l\}\). The indices \(k', l' \in \{1, \ldots, |a| + |b| - 2\}\) correspond to \(a_k, a_l\) in \(a \setminus \{a_k, b_l\}\).

\(7)\ \circ\ \text{followed by} \ \xi: \ \text{The diagram}

\[
P(p, a) \otimes P(q, b) \xrightarrow{\varphi_{i}} P(p + q, a \circ_i b) \xrightarrow{\xi_{k''}} P(p + q + 1, d)
\]

\[
P(p + q, a \circ_i b) \xrightarrow{\xi_{i', j'}} P(p + q + 1, d')
\]
is commutative whenever it is defined with \(1 \leq i \neq k \leq |a|, 1 \leq j \neq l \leq |b|,
\[
d = (a \setminus \{a_k, b_l\}) \setminus \{a_k, b_l\}, \quad d' = (a \setminus \{a_k, b_l\}) \setminus \{a_i, b_j\},
\]
and \(s \in \Sigma_{|a| + |b| - 2}\) the permutation defined by \(ds = d'\). The indices \(k', l' \in \{1, \ldots, |a| + |b| - 2\}\) correspond to \(a_k, b_l\) in \(a \setminus \{a_k, b_l\}\). Similarly, the indices \(i', j' \in \{1, \ldots, |a| + |b| - 2\}\) correspond to \(a_i, b_j\) in \(a \setminus \{a_k, b_l\}\).

**Proof.** Suppose given a \(C\)-colored modular operad \((P, \gamma^P)\) as in Prop. 13.2.6

1. Its equivariant structure is given by the maps \(\gamma^P_{C, \sigma}\) with \(C \sigma\) a permuted corolla (Example 13.1.9) as in Example 13.2.8(1).

2. The structure map \(\varphi_{j}\) is the composite

\[
P(p, a) \otimes P(q, b) \xrightarrow{\varphi_{i}} P(p + q, a \circ_i b) \xrightarrow{\gamma^P_{T_{i,j}}(\pi, b)} P(p, a) \otimes P(q, b)
\]

with \(T\) a barbell (Example 13.1.10) and \(\gamma^P_{T}\) as in Example 13.2.8(2).

3. The structure maps \(\xi_{i}^j\) are the maps \(\gamma^P_{C}\) with \(\xi_{i}^j C\) a contracted corolla (Example 13.1.11) as in Example 13.2.8(3).

The seven generating axioms are consequences of:

- corresponding facts of graph substitution involving permuted corollas, barbells, and contracted corollas;
- associativity of \(\gamma^P\).

More precisely:

1. The first generating axiom about the commutativity of \(\circ\) is a consequence of the equality

\[
T_{j,a}^{(q, b), (p, a)} = (C \sigma)(T_{i,j}^{(p, a), (q, b)})
\]

in which \(C\) is the \((p + q, a \circ_i b)-\)corolla. This equality says that the only difference between the barbells \(T_{j,a}^{(q, b), (p, a)}\) and \(T_{i,j}^{(p, a), (q, b)}\) is a suitable leg permutation.

2. The second generating axiom about the equivariance of \(\circ\) is a consequence of the equality

\[
T_{\sigma^{-1}(i), j}^{(p, a \sigma), (q, b)}(C_{(p, a) \sigma}) = (C \sigma')(T_{i,j}^{(p, a), (q, b)}),
\]
in which $C$ is the $(p + q, a_i, b_j)$-corolla. This equality says that the only difference between the barbells $T^{(p,a_i\sigma),(q,b_j)}_{\sigma^{-1}(i),j}$ and $T^{(p,a),(q,b)}_{i,j}$ is a suitable leg permutation and a permutation at the vertex with genus-profile $(p, q)$.

(3) The third generating axiom about the associativity of $\circ$ is a consequence of the equality

$$T^{(p,q,a\circ b),(r\circ c)}_{s,l} = T^{(p,a),(q,b)}_{i,j} T^{(r,c),(s,l)}_{T^{(q,b),(r\circ c)}}.$$

This equality says that to create a stable graph of the form

$$\begin{array}{c}
\circ \quad b_j \quad b_k \quad c_l \quad \circ \\
\circ \quad b_j \quad b_k \quad c_l \quad \circ
\end{array}$$

with three vertices and two internal edges with colors $a_i = b_j$ and $b_k = c_l$, where the set of legs is allowed to be empty, as the graph substitution of two barbells, the internal edges can be created in either order.

(4) The fourth generating axiom about the commutativity of $\xi$ comes from the equality

$$\left(\xi^{k\ell'} C_{(p+1,a_i),(a_i)}\right)\left(\xi^{ij} C_{(p,a)}\right) = \left(\xi^{ij'} C_{(p+1,a_i),(a_i)}\right)\left(\xi^{kl} C_{(p,a)}\right).$$

This equality says that to create a bowtie-like stable graph

$$\begin{array}{c}
\circ \quad a_i \quad a_k \quad \circ \\
\circ \quad a_i \quad a_k \quad \circ
\end{array}$$

with two loops of colors $a_i = a_j$ and $a_k = a_l$, as the graph substitution of two contracted corollas, the loops can be created in either order. See Example 13.1.14.

(5) The fifth generating axiom about the equivariance of $\xi$ comes from the equality

$$\left(\xi^{\sigma^{-1}(i)} C_{(p,a_i)}\right)\left(\xi^{ij} C_{(p,a)}\right) = \left(\xi^{ij} C_{(p,a_i)}\right)\left(\xi^{kl} C_{(p,a)}\right).$$

This equality says that a permuted corolla substituted into a contracted corolla can be rewritten as a contracted corolla substituted into a permuted corolla. See Example 13.1.15.

(6) The sixth generating axiom about the commutativity between $\xi$ and $\circ$ comes from the equality

$$T^{(p+1,a\setminus\{a_i\},b_j),(q,b)}_{\xi^{kl'} c_{i,j}} = \left(\xi^{kl'} C_{(p+q,a\setminus\{a_i\})}\right)\left(T^{(p,b),(q,b)}_{i,j}\right).$$

This equality says that a stable graph of the form

$$\begin{array}{c}
\circ \quad a_k \quad a_i \quad b_j \\
\circ \quad a_k \quad a_i \quad b_j
\end{array}$$

with two vertices and two internal edges of colors $a_i = b_j$ and $a_k = a_i$ can be created by either substituting a barbell into a contracted corolla or substituting a contracted corolla into a barbell. See Example 13.1.15.
(7) The seventh generating axiom comes from the equality

\[
\xi' \xi'' C(\sigma(p+q,\varnothing)) \left( T_{k,l}^{(p,q)}(a,b) \right) = \left( C(\sigma(p+q+1,\varnothing)) \sigma \right) \xi' \xi'' C(\sigma(p+q,\varnothing)) \left( T_{k,l}^{(p,q)}(a,b) \right).
\]

This equality says that, up to a permutation of the leg ordering, the two internal edges in a stable graph of the form

![Diagram of a stable graph](image)

with two vertices and two internal edges with colors \(a_i = a_j\) and \(a_k = a_l\) can be created in either order by substituting a barbell into a contracted corolla. See Example 13.1.16.

Conversely, suppose given a tuple \((P, \circ, \xi)\) with \(P \in \mathcal{M}_{\text{St}}^{(N \times \Sigma_2^+)}\) satisfying the seven generating axioms. To show that it is canonically a \(\mathcal{C}\)-colored modular operad, the plan of the proof is the same as in the second half of the proof of Theorem 12.4.2 for the coherence of colored cyclic operads. So it suffices to prove the following two statements:

1. Every stable graph \(G\) has an iterated graph substitution presentation

\[ G = G_n(G_{n-1})\cdots(G_1) \]

with each \(G_i\) a permuted corolla, a barbell, or a contracted corolla. For each \(1 \leq j \leq n-1\), in the \(j\)th layer \(G_j\) is the only possible non-corolla; a corolla is substituted into every other vertex in \(G_{j+1}\). Such a presentation exists by Prop. 13.1.13.

2. Any two such presentations of \(G\) are connected by a finite number of elementary moves. Each elementary move replaces a sub-sequence in a presentation by another sequence with the same graph substitution that corresponds to the equivariant structure or one of the seven generating axioms (13.4.2)–(13.4.8).

Once we know statement (2) is true, we can then define the structure map

\[
\gamma^P_G = \gamma^P_{G_n} \circ \gamma^P_{G_{n-1}} \circ \cdots \circ \gamma^P_{G_1}.
\]

Each map \(\gamma^P_{G_i}\) is an equivariant structure map, some \(\rho_j\), or some \(\xi^{ij}\) depending on whether \(G_i\) is a permuted corolla, a barbell, or a contracted corolla. Associativity and unity is guaranteed by definition, and it is independent of the choice of a presentation by statement (2).

Define a stratified presentation of \(G\) as a presentation

\[ G = G_n(G_{n-1})\cdots(G_{i+1})(G_i)\cdots(G_1) \]

consisting of three consecutive sub-sequences, each of which may be empty:

- Each of \(G_1, \ldots, G_i\) is a barbell.
- Each of \(G_{i+1}, \ldots, G_{n-1}\) is a contracted corolla.
- \(G_n\) a permuted corolla.

To prove statement (2), it suffices to prove the following two statements.
(2a) Every presentation of $G$ is connected to a stratified presentation of $G$ by a finite number of elementary moves.

(2b) Any two stratified presentations of $G$ are connected by a finite number of elementary moves.

To prove statement (2a), suppose given a presentation of $G$. We first use the commutativity of $\circ$ (13.4.2), the equivariance of $\circ$ (13.4.3), and the equivariance of $\xi$ (13.4.6) to move all the permuted corollas to the highest indexed entries. In any later step, if a permuted corolla is created in the process, we will tacitly use these three elementary moves repeatedly to move it to the highest indexed subsequence with only permuted corollas. With this in mind, next we use (13.4.2) and (13.4.7) to move all the contracted corollas past all the barbells. At this stage, our presentation of $G$ looks like

$$G_q \cdots (G_n) (G_{n-1}) \cdots (G_{i+1}) (G_i) \cdots (G_1) \tag{13.4.9}$$

in which each of $G_n, \ldots, G_q$ is a permuted corolla. Finally, using the equivariant structure we can replace this subsequence of permuted corollas by their graph substitution, which is a single permuted corolla. So after finitely many elementary moves, we have a stratified presentation.

To prove statement (2b), suppose given two stratified presentations

$$\mathcal{H} = G_n (G_{n-1}) \cdots (G_{i+1}) (G_i) \cdots (G_1),$$

$$\mathcal{H}' = G_p' (G_{p-1}') \cdots (G_{j+1}') (G_j') \cdots (G_1') \tag{13.4.10}$$

of $G$ with $G_n$ and $G_p'$ permuted corollas. In $\mathcal{H}$ each $G_k$ for $1 \leq k \leq n - 1$ creates one internal edge in $G$, so $G$ has exactly $n - 1$ internal edges. Since the same is true for $\mathcal{H}'$, we conclude that $n = p$. Moreover, since a permuted/contracted corolla has only one vertex, the graph substitution

$$G_i \cdots (G_1) \tag{13.4.11}$$

has all the vertices in $G$ with the correct genus-profile at each vertex and is simply-connected because barbells are simply-connected. Since the same is true in $\mathcal{H}'$, we conclude that $i = j$.

We call $e_1$ the internal edge in the barbell $G_1$. Since $e_1$ is an internal edge in $G$ connecting two distinct vertices, in $\mathcal{H}'$ it is created in $G_{i+1}'$ for some $1 \leq l \leq n - 1$.

- If $e_1$ is created among the barbells $G_i'$ for some $1 \leq l \leq i$, then we can use the elementary moves corresponding to the associativity of $\circ$ (13.4.4) to replace $\mathcal{H}'$ by a stratified presentation whose lowest indexed entry creates $e_1$.
- If $e_1$ is created among the contracted corollas $G_i'$ for some $i + 1 \leq l \leq n - 1$, then we first use the commutativity of $\xi$ (13.4.5) repeatedly to move it down to the $(i+1)$st entry, which is the lowest indexed contracted corolla. Then we use (13.4.4) multiple times and (13.4.8) once to move it further down to the lowest indexed entry. In using (13.4.8) a permuted corolla is created, so our presentation is momentarily not stratified. We move this permuted corolla to the second highest indexed entry and compose with the existing permuted corolla as we did in the proof of (2a) above. In each
step below, if a permuted corolla is created, then we will tacitly perform these steps.

So after finitely many elementary moves starting with the original $H'$, we have a stratified presentation whose lowest indexed entry creates $e_1$. Using (13.4.2) if necessary, we may therefore assume that $G_1 = G'_1$.

Next we repeat the steps in the previous paragraph for the internal edge $e_2$ in $G_2$, and so forth including the internal edge $e_i$ in $G_i$. After these finitely many elementary moves starting with $H'$, we may assume that $G_k = G'_k$ for $1 \leq k \leq i$. The iterated contracted corollas

$$H = G_{n-1} \cdots (G_{i+1}) \quad \text{and} \quad H' = G'_{n-1} \cdots (G'_{i+1})$$

create the same subset of internal edges in $G$. We can use the elementary moves corresponding to (13.4.5) repeatedly to connect them, so we may further assume that $G_l = G'_l$ for $1 \leq l \leq n - 1$.

Finally, each of the permuted corollas $G_n$ and $G'_n$ simply corrects the leg ordering from that of the graph substitution $G_{n-1} \cdots (G_1)$ to that of $G$. So we conclude that $G_n = G'_n$, thereby proving (2b). □

**Remark 13.4.10.** The one-colored case of Theorem 3.7 in [13.4] is close to Theorem 3.7. However, there is a minor oversight in Lemma 3.4 and Theorem 3.7, as pointed out in page 125. The issue is that, when a modular operad is restricted to just the structure maps for simply-connected stable graphs, the resulting structure is still not a non-unital cyclic operad. The monad for cyclic operads is parametrized by unrooted trees, which by definition have non-empty sets of legs. However, a stable graph, simply-connected or not, is allowed to have an empty set of legs. For example, an isolated vertex $\bullet$, whose vertex has genus $> 1$, and the barbell $\bullet - \bullet$, where each vertex is given a positive genus, are both legless simply-connected stable graphs, but not unrooted trees. So modular operads have structure maps corresponding to such stable graphs, but cyclic operads do not.

**Corollary 13.4.11.** A map $f : P \rightarrow Q$ of $\mathcal{C}$-colored modular operads is exactly a map of equivariant objects

$$f : P \rightarrow Q \in \mathcal{M}^{\text{St}(N \times \Sigma^p)}$$

such that:

1. The diagram

$$\begin{array}{ccc}
P(p, a) \otimes P(q, b) & \xrightarrow{\circ_j} & Q(p, a) \otimes Q(q, b) \\
& \downarrow{\circ_j} & \downarrow{\circ_j} \\
P(p + q, a \circ_j b) & \xrightarrow{f} & Q(p + q, a \circ_j b)
\end{array}$$

is commutative whenever it is defined.
(2) The diagram

\[
\begin{array}{ccc}
P(p,c) & \xrightarrow{f} & Q(p,c) \\
\xi^{ij} \downarrow & & \downarrow \xi^{ij} \\
P(p+1,c \setminus \{c_i,c_j\}) & \xrightarrow{f} & Q(p+1,c \setminus \{c_i,c_j\})
\end{array}
\]

is commutative whenever it is defined.

Proof. By Prop. 13.2.9 a map of colored modular operads is an entrywise map that preserves all the structure maps \(\gamma_G\) for \((p,c) \in \text{St}(\mathbb{N}, \mathcal{C})\) and \(G \in \text{StGr}(p,\mathcal{C})\). By the decomposed form of \(\gamma_G\) in (13.4.9), it is enough for \(f\) to preserve the equivariant structure and the operations \(\rho_j\) and \(\xi^{ij}\).

13.5. Applications: Cofibrant Resolutions of Modular Operads

Let us now provide some illustrations of the cofibrant resolution in Theorem 13.3.4(4) and the coherence of modular operads in Theorem 13.4.1.

Example 13.5.1 (Deligne-Grothendieck-Knudsen moduli spaces). Consider the setting of Example 12.5.1 with \(\mathcal{M}_{g,n}\) the Deligne-Grothendieck-Knudsen moduli space of stable \(n\)-pointed curves of genus \(g\). The family of spaces

\[
\mathcal{M} = \{\mathcal{M}_{g,n}\}
\]

with \((g,n)\) stable pairs (i.e., \(2(g-1)+n>0\)) is a one-colored topological modular operad \([GK98]\) (6.2). Its equivariant structure is given by permutation of marked points. The operations \(\rho_j\) are given by gluing of curves along the indicated marked points. Each of the maps \(\xi^{ij}\) glues the \(i\)th and the \(j\)th marked points of a given curve \([GK94]\) (1.4.4).

Since the \(\Sigma\)-action permutes the marked points, \(\mathcal{M}\) is \(\Sigma\)-cofibrant. Therefore, Theorem 13.3.4(4) applies to yield a cofibrant resolution

\[
W^{\text{mod}}([0,1],\mathcal{M}) \xrightarrow{\eta} \mathcal{M}
\]

of the one-colored topological modular operad \(\mathcal{M}\).

Example 13.5.2 (Riemann surfaces with holes). Still with \(M = \text{Top}\) and \(J = ([0,1],\text{max})\), for a stable pair \((g,n)\) suppose \(\mathcal{M}(g,n)\) is the moduli space of Riemann surfaces with genus \(g\) and \(n\) parametrized holes, i.e., complex analytic embeddings

\[
\prod_{k=1}^n D \xrightarrow{\phi} S
\]

in which \(D\) is the closed unit disk in the complex plane. Similar to Example 12.5.2 the family of spaces

\[
\mathcal{M} = \{\mathcal{M}(g,n)\}
\]

with \((g,n)\) stable pairs is a one-colored topological modular operad \([Mar08]\) (Section 7). Its equivariant structure is given by permutation of the parametrized holes. The structure maps \(\rho_j\) are induced by sewing Riemann surfaces along the indicated parametrized holes. Each structure map \(\xi^{ij}\) sews the \(i\)th and the \(j\)th parametrized holes of the same Riemann surface.
Since the $\Sigma$-action permutes the parametrized holes, $\mathcal{M}$ is $\Sigma$-cofibrant. Therefore, Theorem 13.3.4(4) applies to yield a cofibrant resolution
\[ W^{\text{mod}}([0,1], \mathcal{M}) \xrightarrow{\eta} \mathcal{M} \]
of the one-colored topological modular operad $\mathcal{M}$.

**Example 13.5.3 (Riemann surfaces with colored holes).** As in Example 12.5.3, there is an $\mathbb{R}_+$-colored topological modular operad $\mathcal{M}^{\mathbb{R}_+}$ that is the $\mathbb{R}_+$-colored version of the one-colored topological modular operad $\mathcal{M}$ in Example 13.5.2. In $\mathcal{M}^{\mathbb{R}_+}$ each copy of the closed unit disk is replaced by any closed disk $D_r$ with radius $r > 0$. The sewing operations $\rho_j$ and $\xi_{ij}$ are now only defined if the two relevant disks have the same radius. Once again Theorem 13.3.4(4) yields a cofibrant resolution
\[ W^{\text{mod}}([0,1], \mathcal{M}^{\mathbb{R}_+}) \xrightarrow{\eta} \mathcal{M}^{\mathbb{R}_+} \]
of the $\mathbb{R}_+$-colored topological modular operad $\mathcal{M}^{\mathbb{R}_+}$.

**Example 13.5.4 (Chain modular operads).** As in Example 12.5.4, suppose $\mathcal{M} = \text{Ch}(k)$ is the cofibrantly generated monoidal model category of chain complexes over a field $k$ of characteristic zero equipped with the commutative interval $J = N\Delta^1$. Every object in $\text{Ch}(k)^{\text{St}(\mathbb{N} \times \Sigma^\infty_+)}$ is cofibrant. Therefore, Theorem 13.3.4(4) applies to yield a cofibrant resolution
\[ W^{\text{mod}}(N\Delta^1, P) \xrightarrow{\eta} P \]
for every $\mathcal{C}$-colored modular operad $P$ in $\text{Ch}(k)$.

For example, suppose $C_\bullet$ denotes the singular chain functor with coefficients in $k$. Then we have the cofibrant resolutions
\[
W^{\text{mod}}(N\Delta^1, C_\bullet \mathcal{M}) \xrightarrow{\eta} C_\bullet \mathcal{M},
\]
\[
W^{\text{mod}}(N\Delta^1, C_\bullet \mathcal{M}^{\mathbb{R}_+}) \xrightarrow{\eta} C_\bullet \mathcal{M}^{\mathbb{R}_+},
\]
of the chain images of the (colored) topological modular operads in Examples 13.5.1, 13.5.2, and 13.5.3.
Bibliography


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[Lor∞] F. Loregian, This is the (co)end, my only (co)friend, preprint available at https://arxiv.org/abs/1501.02503.


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