PM4SAND (VERSION 2):  
A SAND PLASTICITY MODEL FOR EARTHQUAKE ENGINEERING APPLICATIONS

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May 2012
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Report No. UCD/CGM-12/01

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May 2012
ABSTRACT

The sand plasticity model PM4Sand (version 2) for geotechnical earthquake engineering applications is presented. The model follows the basic framework of the stress-ratio controlled, critical state compatible, bounding surface plasticity model for sand presented by Dafalias and Manzari (2004). Modifications to the model were developed and implemented by Boulanger (2010, version 1) and further herein (version 2) to improve its ability to approximate the stress-strain responses important to geotechnical earthquake engineering applications; in essence, the model was calibrated at the equation level to provide for better approximation of the trends observed across a set of experimentally- and case history-based design correlations. These constitutive modifications included: revising the fabric formation/destruction to depend on plastic shear rather than plastic volumetric strains; adding fabric history and cumulative fabric formation terms; modifying the plastic modulus relationship and making it dependent on fabric; modifying the dilatancy relationships to provide more distinct control of volumetric contraction versus expansion behavior; providing a constraint on the dilatancy during volumetric expansion so that it is consistent with Bolton’s (1986) dilatancy relationship; modifying the elastic modulus relationship to include dependence on stress ratio and fabric history; modifying the logic for tracking previous initial back-stress ratios (i.e., loading history effect); recasting the critical state framework to be in terms of a relative state parameter index; simplifying the formulation by restraining it to plane strain without Lode angle dependency for the bounding and dilation surfaces; and providing default values for all but three primary input parameters. Version 2 of the model includes an improved numerical implementation and coding as a user defined material in a dynamic link library (DLL) for use with the commercial program FLAC (Itasca 2011). The numerical implementation and DLL module are described. The behavior of the model is illustrated by simulations of element loading tests covering a broad range of conditions, including drained and undrained, cyclic and monotonic loading under a range of initial confining and shear stress conditions, which can then be compared to typical design relationships. The model is shown to provide reasonable approximations of desired behaviors and to be relatively easy to calibrate.
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1. INTRODUCTION

Nonlinear deformation analyses for problems involving liquefaction are increasingly common in earthquake engineering practice. Constitutive models for sand that have been used in practice range from relatively simplified, uncoupled cycle-counting models to more complex plasticity models (e.g., Dawson et al. 2001, Byrne et al. 2004, Wang et al. 1990, Papadimitriou et al. 2001, Yang et al. 2003, Dafalias and Manzari 2004). Each constitutive model has certain advantages and limitations that can be illustrated for potential users by documents showing the constitutive response of the model in element tests that cover a broad range of the conditions that may be important to various applications in practice (e.g., Beaty 2009).

The information available for calibration of constitutive models in design practice most commonly include basic classification index tests (e.g., grain size distributions), penetration resistances (e.g., SPT or CPT), and shear wave velocity ($V_s$) measurements. More detailed laboratory tests, such as triaxial or direct simple shear tests, are almost never available due to the problems with overcoming sampling disturbance effects and the challenge of identifying representative samples from highly heterogeneous deposits.

Constitutive models for geotechnical earthquake engineering applications must be able to approximate a broad mix of conditions in the field. For example, a single geotechnical structure like the schematic earth dam shown in Figure 1.1 can have strata or zones of sand ranging from very loose to dense under a wide range of confining stresses, initial static shear stresses (e.g., at different points beneath the slope), drainage conditions (e.g., above and below the water table), and loading conditions (e.g., various levels of shaking). The engineering effort is greatly reduced if the constitutive model can reasonably approximate the predicted stress-strain behaviors under all these different conditions. If the model cannot approximate the trends across all these conditions, then extra engineering effort is required in deciding what behaviors should be prioritized in the calibration process, and sometimes by the need to repeat the calibrations for the effects of different initial stress conditions within the same geotechnical structure.

The PM4Sand (version 2) plasticity model for geotechnical earthquake engineering applications is presented herein. The PM4Sand model follows the basic framework of the stress-ratio controlled, critical state compatible, bounding surface plasticity model for sand initially presented by Manzari and Dafalias (1997) and later extended by Dafalias and Manzari (2004). Modifications to the Dafalias-Manzari model were developed and implemented by Boulangier (2010, version 1) and further herein (version 2) to improve its ability to approximate engineering design relationships that are used to estimate the stress-strain behaviors that are important to predicting liquefaction-induced ground deformations during earthquakes. The current version also incorporates an improved numerical implementation and coding as a dynamic link library (DLL) for use with the commercial program FLAC (Itasca 2011).
It is unlikely that any one model can be developed or calibrated to simultaneously fit a full set of applicable design correlations for monotonic and cyclic, drained and undrained behaviors of sand, in part because the various design correlations are not necessarily physically consistent with each other; e.g., they may include a mix of laboratory test-based and case history-based relationships, or they have been empirically derived from laboratory data sets for different sands. Nonetheless, it is desirable that a model, after calibration to the design relationship that is of primary importance to a specific project, be able to produce behaviors that are reasonably consistent with the general magnitudes and trends in other applicable design correlations or typical experimental data.

Stress-strain behaviors of sand that are most commonly the focus in design are listed below, along with reference to a figure showing an example design correlation or typical experimental test result.

- The cyclic resistance ratio (CRR) against triggering of liquefaction, which is commonly estimated based on SPT and CPT penetration resistances with case-history-based liquefaction correlations (e.g., Figure 1.2). The CRR is the cyclic stress ratio (e.g., $\text{CSR} = \tau_{\text{cyc}}/\sigma_{\text{vc}}$, with $\tau_{\text{cyc}}$ = horizontal cyclic shear stress, $\sigma_{\text{vc}}$=vertical consolidation stress) that is required to trigger liquefaction in a specified number of equivalent uniform loading cycles.
- The response under the irregular cyclic loading histories produced by earthquakes, which is approximately represented by the relationship between CRR and number of equivalent uniform loading cycles (e.g., Figure 1.). This aspect of behavior also directly relates to the magnitude scaling factors (MSF) that are used with liquefaction correlations in practice.
- The dependence of CRR on effective confining stresses and sustained static shear stresses. These aspects of behavior are represented by the $K_\sigma$ (Figure 1.) and $K_\alpha$ (Figure 1.5) correction factors, respectively, that are used with liquefaction correlations in practice.
- The accumulation of shear strains after triggering of liquefaction. Evaluations of reasonable behavior are often based on comparisons to laboratory tests results for similar soils in the literature (e.g., Figure 1.6).
- The strength loss as a consequence of liquefaction, which may involve explicitly modeling phenomena such as void redistribution or empirically accounting for it through case history-based residual strength correlations (e.g., Figure 1.7).
- The small-strain shear modulus which can be obtained through in-situ shear wave velocity measurements.
- The shear modulus reduction and equivalent damping ratio relationships prior to triggering of liquefaction. These aspects of behavior are commonly estimated using empirical correlations derived from laboratory test results for similar soils in the literature (e.g., Figure 1.8).
- Drained monotonic shear strengths and stress-strain behavior (e.g., Figure 1.9). Peak friction angles may be estimated using relationships such as Bolton's (1986) relative dilatancy index, $I_R$ (Figure 1.10) or correlations to SPT and CPT penetration resistances.
- Undrained monotonic shear strengths and stress-strain behavior (e.g., Figure 1.11), which may be estimated using correlations to SPT and CPT penetration resistances.
- The volumetric strains during drained cyclic loading (Figure 1.12 and Figure 1.13) or due to reconsolidation following triggering of liquefaction (e.g., Figure 1.14), both of which may be estimated using empirical correlations derived from laboratory test results for similar soils in the
The constitutive model described herein was developed for earthquake engineering applications, with specific goals being: (1) the ability to reasonably approximate empirical correlations used in practice, and (2) an ability to be calibrated within a reasonable amount of engineering effort. In essence, the approach taken was to calibrate the constitutive model at the equation level, such that the functional forms for the various constitutive relationships were chosen for their ability to approximate the important trends embodied in the extensive laboratory-based and case history-based empirical correlations that are commonly used in practice.

The organization of this report is structured as follows.

- Section 2 of this report contains a description of the model formulation.
- Section 3 contains a description of the model's implementation as a user-defined material in a dynamic link library for use in the commercial program FLAC (Itasca 2011).
- Section 4 of this report contains a summary of the model input parameters, guidance on model parameter selections, and then illustrations of the model responses to a broad range of elemental loading conditions.
- Section 5 contains summary remarks regarding the model and its use in practice.

Revisions to PM4Sand in version 2 include: a new numerical implementation for working with the subzoning procedure and mixed discretization scheme of FLAC; implementation as a dynamic link library (DLL); clarifications and corrections to the manual; minor corrections to the constitutive algorithm; and re-calibration of the model, resulting in changes to the default values for some secondary parameters. The simulations presented in this report were prepared using the DLL module `modelpm4s003.dll` compiled on May 15, 2012; note that the compilation date is included in the properties of the dll file.
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2. MODEL FORMULATION

The sand plasticity model presented herein follows the basic framework of the stress-ratio controlled, critical state compatible, bounding-surface plasticity model for sand presented by Dafalias and Manzari (2004). The Dafalias and Manzari (2004) model extended the previous work by Manzari and Dafalias (1997) by adding a fabric-dilatancy related tensor quantity to account for the effect of fabric changes during loading. The fabric-dilatancy related tensor was used to macroscopically model the effect that microscopically-observed changes in sand fabric during plastic dilation have on the contractive response upon reversal of loading direction. Dafalias and Manzari (2004) provide a detailed description of the motivation for the model framework, beginning with a triaxial formulation that simplifies its presentation and then followed by a multi-axial formulation. The model proposed herein is presented in its multi-axial formulation, along with the original framework of the Dafalias-Manzari model for comparison.

2.1 Basic stress and strain terms

The basic stress and strain terms for the proposed model are as follows. The model is based on effective stresses, with the conventional prime symbol dropped from the stress terms for convenience because all stresses are effective for the model. The stresses are represented by the tensor $\sigma$, the principal effective stresses $\sigma_1$, $\sigma_2$, and $\sigma_3$, the mean effective stress $p$, the deviatoric stress tensor $s$, and the deviatoric stress ratio tensor $r$. The present implementation was further simplified by casting the various equations and relationships in terms of the in-plane stresses only. This limits the present implementation to plane-strain applications and is not correct for general cases, but it has the advantage of simplifying the implementation and improving computational speed by reducing the number of operations. Expanding the implementation to include the general case should not affect the general features of the model. Consequently, the relationships between the various stress terms can be summarized as follows:

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix}$$ (1)

$$p = \frac{\sigma_{xx} + \sigma_{yy}}{2}$$ (2)

$$s = \sigma - pI = \begin{pmatrix} s_{xx} & s_{xy} \\ s_{xy} & s_{yy} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} - p & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} - p \end{pmatrix}$$ (3)
\[\mathbf{r} = \frac{\mathbf{s}}{p} = \begin{pmatrix} r_{xx} & r_{xy} \\ r_{xy} & r_{yy} \end{pmatrix} = \begin{pmatrix} \frac{\sigma_{xx} - p}{p} & \frac{\sigma_{xy}}{p} \\ \frac{\sigma_{xy}}{p} & \frac{\sigma_{yy} - p}{p} \end{pmatrix}\] (4)

Note that the deviatoric stress and deviatoric stress ratio tensors are symmetric with \(r_{xx} = -r_{yy}\) and \(s_{xx} = -s_{yy}\) (meaning a zero trace), and that \(\mathbf{I}\) is the identity matrix.

The model strains are represented by a tensor \(\mathbf{\varepsilon}\), which can be separated into the volumetric strain \(\varepsilon_v\) and the deviatoric strain tensor \(\mathbf{e}\). The volumetric strain is,

\[\varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy}\] (5)

and the deviatoric strain tensor is,

\[\mathbf{e} = \mathbf{\varepsilon} - \varepsilon_v \frac{\mathbf{I}}{3} = \begin{pmatrix} \varepsilon_{xx} - \frac{\varepsilon_v}{3} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} - \frac{\varepsilon_v}{3} \end{pmatrix}\] (6)

The deviatoric and volumetric strain terms are separated into elastic and plastic components,

\[d\mathbf{e} = d\mathbf{e}^e + d\mathbf{e}^p\] (7)

\[d\varepsilon_v = d\varepsilon_v^e + d\varepsilon_v^p\] (8)

where
- \(\mathbf{e}^e\) = elastic deviatoric strain tensor
- \(\mathbf{e}^p\) = plastic deviatoric strain tensor
- \(\varepsilon_v^e\) = elastic volumetric strain
- \(\varepsilon_v^p\) = plastic volumetric strain

### 2.2 Critical state

Dafalias and Manzari (2004), based on findings in Li and Wang (1998), used a power relationship to approximate the curving of the critical state line (Schofield and Wroth 1968) that occurs over a broad range of confining stresses,
\[ e_{cs} = e_o - \lambda \left( \frac{p_{cs}}{p_A} \right)^m \]  

(9)

where \( p_{cs} \) = mean stress at critical state, \( e_{cs} \) = critical state void ratio, and \( e_o \), \( \lambda \), and \( m \) are parameters controlling the position and shape of the critical state line. The state of the sand was then described using the state parameter (Been and Jefferies 1985), which is the difference between the current void ratio (\( e \)) and the critical state void ratio (\( e_{cs} \)) at the same mean effective stress (\( p_{cs} \)).

The model proposed herein instead uses the relative state parameter index (\( \xi_R \)) as presented in Boulanger (2003a) and shown in Figure 2.1(a). The relative state parameter (Konrad 1988) is the state parameter normalized by the difference between the maximum void ratio (\( e_{max} \)) and minimum void ratio (\( e_{min} \)) values that are used to define relative density (\( D_R \)). The relative state parameter "index" is just the relative state parameter defined using an empirical relationship for the critical state line. Boulanger (2003a) used Bolton's (1986) dilatancy relationship to define the empirical critical state line and thus arrived at,

\[ \xi_R = D_{R,cs} - D_R \]  

(10)

\[ D_{R,cs} = \frac{R}{Q - \ln \left( \frac{100 p}{p_A} \right)} \]  

(11)

where \( D_{R,cs} \) = relative density at critical state for the current mean effective stress. The parameters \( Q \) and \( R \) were shown by Bolton (1986) to be about 10 and 1.0, respectively, for quartzitic sands. Critical state lines using the above expression with \( Q \) values of 9 and 10 with \( R \) values of 1.0 and 1.5 are shown in Figure 2.1.(b).

2.3 Bounding, dilation, and critical surfaces

The model incorporates bounding, dilation, and critical surfaces following the form of Dafalias and Manzari (2004). The present model simplifies the surfaces by removing the Lode angle dependency (e.g., friction angles are the same for compression or extension loading) that was included in the Dafalias-Manzari model, such that the bounding (\( M^b \)) and dilation (\( M^d \)) ratios can be related to the critical stress (\( M \)) ratio by the following simpler expressions.

\[ M^b = M \cdot \exp \left( -n^b \xi_R \right) \]  

(12)

\[ M^d = M \cdot \exp \left( n^d \xi_R \right) \]  

(13)
where \( n^b \) and \( n^d \) are parameters determining the values of \( M^b \) and \( M^d \), respectively. For the present implementation, the mean normal stress \( p \) is taken as the average of the in-plane normal stresses, \( q \) is the difference in the major and minor principal in-plane stresses, and the relationship for \( M \) is therefore reduced to

\[
M = 2 \cdot \sin(\phi_{cv})
\]

(14)

where \( \phi_{cv} \) is the constant volume or critical state effective friction angle. The three surfaces can, for the simplifying assumptions described above, be conveniently visualized as linear lines on a q-p plot (where \( q = \sigma_1 - \sigma_3 \)) as shown in Figure 2.2 or as circular surfaces on a stress-ratio graph of \( r_{yy} \) versus \( r_{xy} \) as shown in Figure 2.3.

As the model is sheared toward critical state (\( \xi_R = 0 \)), the values of \( M^b \) and \( M^d \) will both approach the value of \( M \). Thus the bounding and dilation surfaces move together during shearing until they coincide with the critical state surface when the soil has reached critical state.

The above functional form for the bounding stress ratio controls the relationship between peak friction angle and relative state, which is consistent with the forms and data previously proposed by Been and Jeffries (1985) and Konrad (1986). The data from those studies were primarily for sands that were dense of critical, and the above relationship can reasonable fit those data. The few data points for loose-of-critical sands show that the peak friction angles (presumably determined at the limit of strains possible within the laboratory tests) were only slightly smaller than the critical state values, such that extending the above relationships to loose-of-critical sands may tend to underestimate the peak friction angles. Consequently, the present model allows \( n^b \) and \( n^d \) to be different for loose-of-critical and dense-of-critical states for the same sand.

### 2.4 Yield surface and image back-stress ratio tensors

The yield surface and image back-stress ratio tensor \( \alpha \) follow those of the Dafalias-Manzari model, although their final form is considerably simplified by the prior assumption of removing any Lode angle dependency. The yield surface is a small cone in stress space, and is defined in stress terms by the following expression:

\[
f = \left[ (s - p\alpha) : (s - p\alpha) \right]^{1/2} - \sqrt{1/2}pm = 0
\]

(15)

The back-stress ratio tensor \( \alpha \) defines the center of the yield surface, and the parameter \( m \) defines the diameter of the cone in terms of stress ratio. The yield function can be rewritten to emphasize the role of stress ratio terms as follows,

\[
f = \sqrt{(r - \alpha) : (r - \alpha)} - \sqrt{1/2}m
\]

(16)

The yield function can then be visualized as related to the distance between the stress ratio \( r \) and the back-stress ratio \( \alpha \), as illustrated in Figure 2.3.
The bounding surface formulation now requires that bounding and dilation stress ratio tensors be defined. Dafalias and Manzari (2004) showed that it is more convenient to track back-stress ratios and to similarly define bounding and dilation surfaces in terms of back-stress ratios. An image back-stress ratio tensor for the bounding surface ($\alpha^b$) is defined as,

$$\alpha^b = \sqrt{\frac{1}{2}} [M^b - m] n$$  \hspace{1cm} (17)$$

where the tensor $n$ is normal to the yield surface. An image back-stress ratio tensor for the dilation surface ($\alpha^d$) is similarly defined as,

$$\alpha^d = \sqrt{\frac{1}{2}} [M^d - m] n$$ \hspace{1cm} (18)$$

The computation of constitutive responses can now be more conveniently expressed in terms of back-stress ratios rather than in terms of stress ratios, as noted by Dafalias and Manzari (2004).

2.5 Stress reversal and initial back-stress ratio tensors

The bounding surface formulation, as described in Dafalias (1986) and adopted by Dafalias and Manzari (2004), keeps track of the initial back-stress ratio ($\alpha_{in}$) and uses it in the computation of the plastic modulus. This tracking of one instance in loading history is essentially a first-order method for tracking loading history. A reversal in loading direction is then identified, following traditional bounding surface practice, whenever

$$(\alpha - \alpha_{in}) : n < 0$$ \hspace{1cm} (19)$$

A reversal causes the current stress ratio to become the initial stress ratio for subsequent loading. Small cycles of load reversal can reset the initial stress ratio and cause the plastic modulus to increase accordingly, in which case the stress-strain response becomes overly stiff after a small load reversal. This is a well-known problem in bounding surface formulations for which various approaches offer different advantages and disadvantages.

The model proposed herein tracks both an initial back-stress ratio and a previous initial back-stress ratio ($\alpha_{in}^p$), as illustrated in Figure 2.4. When a reversal occurs, the previous initial back-stress ratio is updated to the initial back stress ratio, and the initial back-stress ratio is updated to the current back-stress ratio, subject to the following constraint. For loading in a positive direction, the initial back-stress ratio is assigned the greater of: (a) the minimum value that the initial back-stress ratio had ever been assigned and (b) zero. For loading in a negative direction, the logic is reversed. In this manner, small amplitude reversals after much stronger loading cannot produce an overly stiff response.
2.6 Elastic strains and moduli

The elastic deviatoric strain and elastic volumetric strain are computed as:

\[ d\varepsilon^e = \frac{ds}{2G} \quad \text{(20)} \]

\[ d\varepsilon^v = \frac{dp}{K} \quad \text{(21)} \]

where \( G \) is the elastic shear modulus and \( K \) is the elastic bulk modulus. The elastic shear modulus in the model proposed herein is dependent on the mean effective stress according to,

\[ G = G_o p_A \left( \frac{p}{p_A} \right)^{1/2} C_{SR} \quad \text{(22)} \]

where \( G_o \) is a constant, \( p_A \) is atmospheric pressure (101.3 kPa), and \( C_{SR} \) is a factor that accounts for stress ratio effects (described below).

Dafalias and Manzari (2004) had included dependence of \( G \) on void ratio following the form of Richart et al. (1970). This aspect was not included in the model herein because: (1) the effects of void ratio changes on \( G \) are small relative to those of confining stress, (2) the value of \( G_o \) is more strongly affected by environmental factors such as cementation and ageing, and (3) the calibration of \( G \) to in-situ shear wave velocity data requires only one constant (\( G_o \)) rather than two (\( G_o \) and \( e \)).

Yu and Richart (1984) showed that the small-strain elastic shear modulus of sand is dependent on the stress ratio and stress ratio history. The effect of stress ratio was shown to generally be less than about 10% when the ratio of major to minor principal effective stresses is less than about 2.5, but to also increase to about 20-30% at higher principal stress ratios. They also showed that stress ratio history caused a reduction in the small-strain elastic shear modulus when the maximum previous stress ratio was greater than the current stress ratio. The effect of stress ratio and stress ratio history on the elastic shear modulus was approximately accounted for in the present model by the factor \( C_{SR} \). The following equation for \( C_{SR} \) is similar in form to that used by Yu and Richart (1984) to represent stress ratio effects, except that it uses stress ratio terms consistent with the present model,

\[ C_{SR} = 1 - C_{SR,0} \left( \frac{M}{M^b} \right)^{m_{SR}} \quad \text{(23)} \]

The above equation approximates Yu and Richart's (1984) results for stress ratio effects when \( C_{SR,0} = 0.3 \) and \( m_{SR} = 2 \). The effects of stress ratio history would cause further reductions, and is more complicated to represent. The calibration examples presented later in this report worked well with \( C_{SR,0} = 0.6 \) and \( m_{SR} = 4 \), which keeps the effect of stress ratio on elastic modulus small at small stress ratios, but lets the effect increase to a 60% reduction when the stress ratio is on the bounding surface.

The elastic bulk modulus is related to the shear modulus through the Poisson's ratio as,


\[
K = \frac{2(1+\nu)}{3(1-2\nu)}G
\]

(24)

as was done by Dafalias and Manzari (2004).

### 2.7 Plastic components without fabric effects

**Loading index**

The loading index (L) is used to compute the plastic component of volumetric strain and the plastic deviatoric strain tensor as,

\[
d\varepsilon_v^p = \langle L \rangle D
\]

(25)

\[
d\varepsilon^p = \langle L \rangle R'
\]

(26)

where D is the dilatancy, R is the direction of d\varepsilon^p, R' is the deviatoric component of R, and \langle \rangle are MacCauley brackets that set negative values to zero [i.e., \langle L \rangle = L if L \geq 0, and \langle L \rangle = 0 if L < 0]. The tensor R for the assumption of no Lode angle dependency is,

\[
R = n + \frac{1}{3} Di
\]

(27)

where n is the unit normal to the yield surface (Figure 2.3). Note that the assumption of no Lode angle dependency also means that R' = n. The dilatancy D relates the incremental plastic volumetric strain to the incremental plastic deviatoric strain,

\[
D = \frac{d\varepsilon_v^p}{|d\varepsilon^p|}
\]

(28)

The dilatancy D can be also related to the conventional engineering shear strain in this plane strain approximation, as

\[
D = \frac{d\varepsilon_v^p}{\sqrt{1/2 |d\gamma^p|}}
\]

(29)

The loading index, as derived in Dafalias and Manzari (2004) is,
\[ L = \frac{1}{K_p} \frac{\partial f}{\partial \sigma} : d\sigma = \frac{1}{K_p} [n : ds - n : rdp] \]

\[ L = \frac{2Gn : de - n : rKde}{K_p + 2G - KKn : r} \quad (30) \]

The stress increment for an imposed increment of strains can then be computed as,

\[ d\sigma = 2Gde + Kde, - (L)(2Gn + KDI) \quad (31) \]

**Hardening and the update of the back-stress ratio**

Updating of the back-stress ratio is dependent on the hardening aspects of the model. Dafalias and Manzari (2004) updated the back-stress ratio according to bounding surface practice as,

\[ d\alpha = (L)\left(\frac{2}{3}\right)h(\alpha^b - \alpha) \quad (32) \]

where \( h \) is the hardening coefficient. The factor of \( 2/3 \) was included for convenience so that model constants would be the same in triaxial and multi-axial derivations. They subsequently showed that the consistency condition \( \delta f = 0 \) was satisfied when the plastic modulus \( K_p \) was related to the hardening coefficient as,

\[ K_p = \frac{2}{3} p \cdot h \cdot (\alpha^b - \alpha) : n \quad (33) \]

This expression can be rearranged so as to show that the consistency equation can be satisfied by expressing the hardening coefficient as,

\[ h = \frac{3}{2} \frac{K_p}{p \cdot (\alpha^b - \alpha) : n} \quad (34) \]

The relationship for the plastic modulus can subsequently take a range of forms, provided that the hardening coefficient and updating of the back-stress ratio follow the above expressions.

**Plastic modulus**

The plastic modulus in the multi-axial generalized form of Dafalias and Manzari (2004), after substituting in their expression for the hardening coefficient, can be expressed as,
where \( h_0 \) and \( C_h \) are scalar parameters and \( e \) is the void ratio. Setting aside the secondary influence of void ratio, this form illustrates that \( K_p \) is proportional to \( G \), proportional to the distance of the back-stress ratio to the bounding back-stress ratio, and inversely proportional to the distance of the back-stress ratio from the initial back-stress ratio.

The plastic modulus relationship was revised in the model presented herein to provide an improved approximation of empirical relationships for secant shear modulus and equivalent damping ratios during drained strain-controlled cyclic loading. The plastic modulus is computed as,

\[
K_p = G \cdot h_0 \cdot \frac{\left(\alpha - \alpha^b\right) : n}{\exp\left(\left[\alpha - \alpha^b\right] : n\right) - 1 + C_{\gamma l}}
\]  

The constant \( C_{\gamma l} \) in the denominator serves to avoid division by zero and has a slight affect on the nonlinearity and damping at small shear strains. If \( C_{\gamma l} = 0 \), then the value of \( K_p \) will be infinite at the start of a loading cycle because \((\alpha - \alpha^b) : n\) will also be zero. In that case, nonlinearity will become noticeable only after \((\alpha - \alpha^b) : n\) becomes large enough to reduce \( K_p \) closer to the value of \( G \) (e.g., \( K_p / G \) closer to 100 or 200). Setting the value of \( C_{\gamma l} = h_0 / 200 \) produces a reasonable response as will be demonstrated later with examples of modulus reduction and equivalent damping ratios. For stress ratios outside the bounding surface [i.e., loose-of-critical states with \((\alpha - \alpha^b) : n < 0\)], the signs in the above expression are modified to allow for negative values of plastic modulus as,

\[
K_p = G \cdot h_0 \cdot \frac{-\left[\alpha - \alpha^b\right] : n}{\exp\left(\left[\alpha - \alpha^b\right] : n\right) - 1 + C_{\gamma l}}
\]  

Plastic volumetric strains - Dilation

Plastic volumetric strains are related to plastic deviatoric strains through the dilatancy \( D \) (Equations 28 and 29), which is computed in the Dafalias and Manzari (2004) model and the base component of the model presented herein (without fabric effects yet) as,

\[
D = A_{do} \cdot \left[\left(\alpha^d - \alpha\right) : n\right]
\]  

Note that dilation (increasing void ratio) occurs whenever the term \((\alpha^d - \alpha) : n\) is less than zero whereas contraction (decreasing void ratio) occurs when it is positive.

The constant \( A_{do} \) in this relationship can be related to the dilatancy relationship proposed by Bolton (1986), which follows from the work of Rowe (1962), through the following sequence of
steps. Bolton showed that the difference between peak and constant volume friction angles could be approximated as,

\[ \phi_{pk} - \phi_{cv} = -0.8\psi \]  

(39)

with

\[ \psi = \tan^{-1}\left( \frac{dc_v^p}{d\gamma^p} \right) \]  

(40)

Since \( \psi \approx \tan(\psi) \) for \( \psi \) less than about 0.35 radians (20 degrees), the difference between peak and constant volume friction angles (in radians) can be approximated as,

\[ \phi_{pk} - \phi_{cv} = -0.8 \frac{dc_v^p}{d\gamma^p} = -0.8\sqrt{ \frac{1}{2} D } \]  

(41)

The peak friction angle is mobilized at the bounding surface, so this can be written as,

\[ \phi_{pk} = \phi_{cv} - 0.8 \sqrt{ \frac{1}{2} A_{do} \left[ (\alpha - \alpha) : n \right] } \]

\[ \phi_{pk} = \phi_{cv} + 0.8 \sqrt{ \frac{1}{2} A_{do} \left[ \left( \frac{M^b - M^d}{\sqrt{2}} \right) : n \right] } \]  

(42)

The term \( n : n \) is equal to unity, and the values of \( \phi_{pk} \) and \( \phi_{cv} \) (again in radians) can be replaced with expressions in terms of \( M^b \) and \( M^d \) as,

\[ \sin^{-1}\left( \frac{M^b}{2} \right) - \sin^{-1}\left( \frac{M}{2} \right) = 0.4A_{do} \left[ M^b - M^d \right] \]  

(43)

This expression can then be rearranged to solve for \( A_{do} \) as,

\[ A_{do} = \frac{1}{0.4} \frac{\sin^{-1}\left( \frac{M^b}{2} \right) - \sin^{-1}\left( \frac{M}{2} \right)}{M^b - M^d} \]  

(44)

where the angles returned by the \( \sin^{-1} \) functions are in radians.

The parameter \( A_{do} \) should thus be chosen to be consistent with the \( n^d \) and \( n^b \) terms that control \( M^b \), and \( M^d \). For example, setting the parameters \( n^b \) and \( n^d \) equal to 0.5 and 0.1, respectively, results in \( A_{do} \) varying from 1.26 for \( \xi_R = -0.1 \) to 1.45 for \( \xi_R = -0.7 \). A default value for \( A_{do} \) is computed based on the above expression using the conditions at the time of model initialization in FLAC (as described in a later section). If an alternative value for \( A_{do} \) is manually input as a property of the model, then the default value will be deactivated.
Alternatively, the stress ratio terms can be replaced with friction angles (in radians) as follows,

\[
\phi_{pk} - \phi_{cv} = 0.4A_{do} \cdot [M^b - M^d]
\]

\[
\phi_{pk} - \phi_{cv} = 0.4A_{do} \cdot [M \exp(-n^b \xi_R) - M(n^d \xi_R)]
\]

\[
\phi_{pk} - \phi_{cv} = 0.4A_{do} \cdot [2 \sin(\phi_{pk}) - 2 \sin(\phi_d)]
\]

\[
\phi_{pk} - \phi_{cv} = 0.8A_{do} \cdot [\sin(\phi_{pk}) - \sin(\phi_d)]
\]

(45)

The sine terms can be replaced with Taylor series, which are quite accurate with just the first two terms as,

\[
sin(\phi) = \phi - \frac{(\phi)^3}{3!}
\]

(46)

Substituting the Taylor series in the above equation gives,

\[
\phi_{pk} - \phi_{cv} = 0.8A_{do} \cdot \left[\left(\frac{\phi_{pk}}{3!}\right)^3 - \left(\frac{\phi_d}{3!}\right)^3\right]
\]

(47)

The parameter \(A_{do}\) can then be solved for as,

\[
A_{do} = \frac{\phi_{pk} - \phi_{cv}}{0.8\left[\phi_{pk} - \phi_d - \frac{\phi_{pk}^3 - \phi_d^3}{6}\right]}
\]

(48)

where the friction angles in the above expression are in radians. This expression provides an alternative view of how the parameter \(A_{do}\) relates to friction angles for a given set of \(n^b\) and \(n^d\) terms that control \(\phi_{pk}\) and \(\phi_d\), respectively. For example, consider the case with the parameters \(n^b\) and \(n^d\) equal to 0.5 and 0.1, respectively, and assuming \(\phi_{cv} = 33\) degrees. For \(\xi_R = -0.1\), we would obtain \(\phi_d = 32.6\) degrees, \(\phi_{pk} = 34.9\) degrees, and \(A_{do} = 1.26\). For \(\xi_R = -0.7\), we would obtain \(\phi_d = 30.5\) degrees, \(\phi_{pk} = 50.6\) degrees, and \(A_{do} = 1.45\).

**Plastic volumetric strains - Contraction**

Plastic volumetric strains during contraction (i.e., whenever \((\alpha^d - \alpha) : n\) is greater than zero) are computed in the Dafalias and Manzari (2004) model using the same expression as used for dilation,

\[
D = A_{do} \cdot [(\alpha^d - \alpha) : n]
\]

(49)
The use of this expression was found in the present study to limit the ability of the model to approximate a number of important loading responses; e.g., it greatly overestimated the slope of the cyclic resistance ratio (CRR) versus number of equivalent uniform loading cycles for undrained cyclic element tests (e.g., Figure 1.).

Plastic volumetric strains during contraction for the model presented herein are computed using the following expression,

\[
D = A_{dc} \left[ (\alpha - \alpha_{in}) : n + C_{in} \right]^2 \frac{(\alpha^d - \alpha) : n}{(\alpha^d - \alpha) : n + C_D} 
\]

\[ A_{dc} = \frac{A_{do}}{h_p} \quad (51) \]

\[ C_{in} = 5.0 \left( -\alpha_{in} : n - \alpha^d : n \right) \quad (52) \]

The various forms in the above relationships were developed to improve different aspects of the calibrated model's performance. The value of \( D \) was set proportional to the square of \(((\alpha-\alpha_{in}) : n + C_{in})\) as this improved the slope of the relationship between CRR and number of uniform loading cycles. The inclusion of the term \( C_{in} \) improves the stress paths for undrained cyclic loading and the volumetric strain response during drained cyclic loading; Inclusion of this constant enables some volumetric strain to develop early in the unloading from a point outside the dilation surface. The constant \( C_{in} \) is zero if the reversal which established \( \alpha_{in} \) was on or below the dilation surface, and increases with increases in the stress ratio at which the last reversal occurred. The remaining terms on the right hand side of the equation were chosen to be close to unity over most of the loading range, while ensuring that \( D \) smoothly goes to zero as \( \alpha \) approaches \( \alpha^d \); Reasonable results were obtained using a \( C_D \) value of 0.10.

The parameter \( A_{dc} \) for contraction was related to the value of \( A_{do} \) for dilation by dividing it by a parameter \( h_p \) that can be varied during the calibration process to obtain desired cyclic resistance ratios. The effect of confining stress on cyclic loading behavior was then conveniently incorporated by making \( h_p \) depend on \( \xi_R \), with the following form chosen so that the model produces results consistent with the design \( K_\sigma \) relationships presented earlier in Figure 1.4.

\[
h_p = h_{po} \cdot \exp \left( -0.7 + 7.0 \left( 0.5 - \xi_R \right)^{2.5} \right) \quad \text{for} \quad \xi_R \leq 0.5 \quad (53) \]

\[
h_p = h_{po} \cdot \exp \left( -0.7 \right) \quad \text{for} \quad \xi_R > 0.5 \quad (54) \]
The constant $h_{po}$ can then be calibrated to arrive at a desired cyclic resistance ratio, once the other
input parameters have been selected.

A limit was imposed on the contraction rate, with the limiting value computed as,

$$D \leq 1.5 \cdot A_{vo} \frac{(\alpha^d - \alpha) : n}{(\alpha^d - \alpha) : n + C_D}$$  \hspace{1cm} (55)$$

This limit avoided numerical issues that were encountered with excessively large contraction
rates. It does not appear to have limited the ability of the model to recreate realistic contraction rates
as illustrated in the calibration examples shown later.

### 2.8 Fabric effects

Dafalias and Manzari (2004) introduced a fabric-dilatancy tensor ($z$) that could be used to account
for the effects of prior straining. Their fabric tensor ($z$) evolved in response to plastic volumetric
dilation strains, according to,

$$dz = -c_z \left( -\sqrt[D]{\epsilon_v^p} \right) (z_{max} n + z)$$  \hspace{1cm} (56)$$

where the parameter $c_z$ controls the rate of evolution and $z_{max}$ is the maximum value that $z$
can attain.

The fabric-dilatancy tensor was modified for the present model as,

$$dz = -\frac{c_z}{1 + \left( \frac{z_{cum}}{2z_{max}} - 1 \right)} \left( -\sqrt[D]{\epsilon_v^p} \right) (z_{max} n + z)$$  \hspace{1cm} (57)$$

In this expression, the tensor $z$ evolves in response to plastic deviatoric strains that occur during
dilation only (i.e., dividing the plastic volumetric strain by the dilatancy gives plastic shear strain). In
addition, the cumulative value of absolute changes in $z$ ($z_{cum}$, a scalar quantity) is computed according
to,

$$dz_{cum} = |dz|$$  \hspace{1cm} (58)$$

The rate of evolution for $z$ therefore decreases with increasing values of $z_{cum}$, which enables the
undrained cyclic stress-strain response to progressively accumulate shear strains rather than lock-up
into a repeating stress-strain loop. In addition, the greatest past peak value (scalar amplitude) for $z$
during its loading history is also tracked.
The values of $z$, $z_{\text{peak}}$, and $z_{\text{cum}}$ are later used to facilitate the accumulation of shear strains under symmetric loading through their effects on the plastic modulus and dilatancy relationships.

The evolution of the fabric tensor terms is illustrated in Figure 2.5 and Figure 2.6 showing the response of a loose sand to undrained cyclic DSS loading without any sustained horizontal shear stress (Figure 2.5) and with a sustained horizontal shear stress (Figure 2.6). These figures show the stress path and stress-strain response of the sand, along with time histories for the back-stress ratios and fabric tensor terms. Note how the fabric terms do not grow until the soil reaches the dilation surface, and how the stress-ratios are limited by the bounding stress ratio. There is no horizontal shear stress reversal for the case shown in Figure 2.6 and thus the back-stress ratio and fabric terms do not reverse either.

**Additional memory of fabric formation history**

Memory of the fabric formation history was included in the model presented herein to improve the ability of the model to account for the effects of sustained static shear stresses and account for differences in fabric effects for various drained versus undrained loading conditions.

The initial fabric tensor ($z_{\text{init}}$) at the start of the current loading path is determined whenever a stress ratio reversal occurs, and thus correspond to the same times that the initial back-stress ratio and previous initial back-stress ratio are updated. The $z_{\text{init}}$ tracks the immediate history terms without any consideration of whether an earlier loading cycle had produced greater degrees of fabric (i.e., the logic is different from that adopted for the updating of back-stress ratio history terms). This history term is used for describing the degree of stress rotation and its effects on plastic modulus, as described later.

Another aspect of the fabric history that is tracked is the mean stress at which the fabric is formed. This aspect of fabric history is tracked by tracking the product of $z$ and $p$, and defining $p_{zp}$ as the mean stress at the time that this product achieves its greatest peak value. The $p_{zp}$ is used in addressing a couple of issues, including the issue of how fabric that is formed during liquefaction may be erased during reconsolidation. For example, saturated sand that develops cyclic mobility behavior during undrained cyclic loading clearly remembers its history of plastic deviatoric strains and then subsequently forgets (to a large extent) this prior strain history when it reconsolidates back to its pre-earthquake confining stress. As another example, the memory of prior strains during undrained cyclic loading is very different than the memory of prior strains during drained cyclic loading. This memory conceptually could be related to the history of plastic and total volumetric strains, but a simpler method to account for this effect is to consider how the mean stress $p$ relates to the value of $p_{zp}$. Conceptually, it appears that prior strain history (or fabric) is most strongly remembered when the soil is operating under mean stresses that are smaller than those that existed when the fabric was formed (i.e., $p \ll p_{zp}$) and then largely forgotten when they are of the same order (i.e., $p \approx p_{zp}$). This attribute will be used in the relationships described later for describing the effects of fabric on dilatancy.

\[
z_{\text{peak}} = \max \left( \frac{z : z}{2}, z_{\text{peak}} \right)
\]
Effect of fabric on plastic modulus

An effect of fabric on the plastic modulus was added to the model presented herein by reducing the plastic modulus as the fabric tensor increased in peak amplitude, as follows:

\[
K_p = G \cdot h_0 \cdot \frac{0.5 \left( \left( \alpha^b - \alpha \right) : n \right)^{0.5}}{\exp \left( \alpha - \alpha_n \right) : n - 1} + C_r \frac{1}{1 + C_K \frac{z_{\text{peak}}}{z_{\text{max}}}} \left( \left( \alpha^b - \alpha \right) : n \right)
\]  

(60)

This reduction in plastic modulus is conceptually motivated by the reduction in modulus that occurs whenever fabric is favorable \((z:n \geq 0)\) and as any cementation is broken down with increasing plastic shear strains. This reduces both the plastic modulus and the hysteretic damping at larger shear strains (note that \(z_{\text{peak}} = 0\) unless the soil has been loaded strongly enough to pass outside the dilation surface), improves the volumetric strains that develop in drained cyclic loading, and improves the path in undrained cyclic loading. Setting \(C_K\) equal to 2.0 was found to produce reasonable responses with particular emphasis on improving (reducing) the equivalent damping ratios at shear strains of 1 to 3% in drained cyclic loading.

Effect of fabric on plastic volumetric dilation

An effect of fabric on the volume change behavior during dilation \((D < 0)\), was added to the model presented herein as follows,

\[
D = A_d \cdot \left( \left( \alpha^d - \alpha \right) : n \right)
\]

(61)

\[
A_d = \frac{A_{do} \left( C_{z_{\text{in2}}} \right)}{\left( \frac{z_{\text{cum}}}{z_{\text{max}}} \right)^3 \left( 1 - \left( \frac{z : n}{\sqrt{2} \cdot z_{\text{peak}}} \right) \right) \left( C_r \right) \left( C_{pzp} \right) \left( C_{p_{\text{min}}} \right) \left( C_{z_{\text{in1}}} \right) + 1}
\]

(62)

\[
C_{pzp} = \frac{1}{1 + \left( \frac{z_{\text{peak}}}{z_{\text{peak}}} \right)^5}
\]

(63)

\[
C_{p_{\text{min}}} = \frac{1}{1 + \left( \frac{p_{\text{min}}}{p} \right)^3}
\]

(64)

\[
C_{z_{\text{in1}}} = 1.0 - \exp \left( -2.0 \frac{z_{\text{in}} : n - z : n}{z_{\text{max}}} \right)
\]

(65)
Consider the six terms added to the denominator. The first term \([z_{\text{cum}}^2/z_{\text{max}}]\) facilitates the progressive growth of strains under symmetric loading by reducing the dilatancy that occurs when a liquefied soil has been sheared through many cycles of loading; note that this term progressively increases with subsequent cycles of loading. The second term facilitates strain-hardening when the plastic shear strain reaches the prior peak value, wherein the term approaches zero (i.e., when \(z:n\) approaches \(z_{\text{peak}}\sqrt{2}\)) and the dilation rate consequently rapidly approaches the virgin loading value of \(A_d\). The third term \(C_{\varepsilon}\) is a calibration constant that can be used to modify the rate of plastic shear strain accumulation. The fourth term \(C_{\varepsilon_{zp}}\) causes the effects of fabric on dilation to be diminished (erased) whenever the current value of \(p\) is near the value of \(p_{zp}\); This term enables the model to provide reasonable predictions of responses to large numbers of either drained or undrained loading cycles. The fifth term \(C_{\text{pmin}}\) provides a minimum amount of shear resistance for a soil after it has temporarily reached an excess pore pressure ratio of 100%; This term is almost zero when \(p'=0\), such that the soil will initially dilate until some minimum \(p'\) has developed, after which the term quickly approaches 1.0. The parameter \(p_{\text{min}}\) is currently set to become equal to 5% of the value of \(p'\) at consolidation (which is the value that exists when the flag FirstCall –see Section 3– was last set equal to 0), with the minimum value of \(p_{\text{min}}\) being 5.0 times the minimum value of \(p'\) (i.e., \(p_{A}/200\)). The sixth term \(C_{z_{\text{in1}}}\) facilitates strain-hardening when stress reversals are not causing fabric changes; i.e., when the initial and current fabric terms are close to equal, the term \(C_{z_{\text{in1}}}\) goes to zero. Lastly, the second term in the numerator, \(C_{z_{\text{in2}}}\), causes the dilatancy to be decreased by up to a factor of 3 under conditions of large strains and full stress (and fabric) reversals, which improves the prediction of cyclic strain accumulation during undrained cyclic loading.

*Effect of fabric on plastic volumetric contraction*

Dafalias and Manzari (2004) used the fabric tensor to modify the dilatancy during contraction \((D > 0)\) as follows,

\[
D = A_{d\varepsilon} \cdot \left[ (\alpha^d - \alpha) : n \right] \left( 1 + (z : n) \right)
\]  

(67)

This relationship enhances the volumetric contraction whenever the fabric is favorable \((z:n \geq 0)\), based on the term \(1 + <z:n>\) as recommended by Dafalias and Manzari (2004).

The effect of fabric on dilatancy during contraction was modified for the present model as,

\[
D = A_{d\varepsilon_{cz}} \cdot \left[ (\alpha - \alpha_n) : n + C_{\varepsilon_{cz}} \right]^2 \frac{(\alpha^d - \alpha) : n}{(\alpha^d - \alpha) : n + C_D}
\]

(68)
\[ A_{dc} = \frac{A_{do}(1 + (z:n))}{h_p C_{dz}} \]  
\[ C_{dz} = \left( 1 - \frac{z_{peak}}{z_{max}} \right) \left( \frac{z_{max}}{z_{max} + z_{cum}} \right) \geq \frac{1}{1 + \frac{z_{max}}{2}} \]  

The denominator term \( C_{dz} \) serves to increase the rate of contraction as \( z_{peak} \) nears \( z_{max} \) or as a large amount of cumulative fabric formation/destruction has taken place. This term was developed for improved modeling of the cyclic strength of denser sands, for which the value of \( h_p \) can be on the order of 100. The degrading of the denominator as \( z_{peak} \) or \( z_{cum} \) increases enables the generation of high excess pore pressures at higher loading levels, and controls the slope of the CRR versus number of uniform loading cycles relationship obtained for undrained element loading. Note that the denominator degrades whether fabric is favorable or not, but that the overall rate of contraction is more enhanced if the fabric is favorable (\( z:n \geq 0 \)). Lastly, the limit on the minimum value of \( C_{dz} \) is required for avoiding division by zero and to avoid over-estimating contraction rates (i.e., small values of \( h_p \) and large values of \( z_{peak} \) or \( z_{cum} \)).

**Effect of fabric on the elastic modulus**

The elastic shear modulus and elastic bulk modulus may degrade with increasing values of cumulative plastic deviator strain term, \( z_{cum} \). This component of the model was added to account for the progressive destruction, with increasing plastic shear strains, of any minor cementation bonds or other ageing- or strain history-related phenomena that produced an increase in small-strain shear modulus. The destruction of minor cementation by plastic shear strains is evidenced in the field by measurements of shear wave velocities in sand that are lower after earthquake shaking than before earthquake shaking (e.g., Arai 2006). The degradation of the elastic shear modulus is computed as,

\[ G = G_o p_A \left( \frac{p}{p_A} \right)^{1/2} C_{SR} \left( 1 + \frac{z_{cum}}{4z_{max}} \right) \left( 1 + \frac{z_{cum}}{4z_{max} C_{DG}} \right) \]

where \( C_{DG} \) is the factor by which the shear modulus is degraded (divided) at very large values of \( z_{cum} \). This change in the elastic shear modulus \( G \) causes the bulk modulus \( K \) to progressively decrease with increasing \( z_{cum} \). The change in \( K \) improves the model's ability to track the stress-strain response of liquefying sand. In particular, decreasing \( K \) with increasing \( z_{cum} \) reduces the rate of strain-hardening after phase transformation at larger shear strain levels, and improves the ability to approximate the hysteretic stress-strain response of a soil as it liquefies.
Effect of fabric on the response under sustained static shear stresses

The formulation of the Dafalias and Manzari (2004) model was found to predict effects of sustained static shear stresses on undrained cyclic loading behavior that are opposite to those observed experimentally; the model predicts that a sustained static shear stress increases the cyclic resistance of loose sands and decreases the cyclic resistance of dense sands. Revisions to the model, as described above, showed that the response under sustained static shear stresses was dependent on several components of the model, such that no one feature controls this aspect of behavior.

The model proposed herein was found to reasonably approximate the effects of a sustained shear stress on the undrained cyclic loading behavior of loose sands, but not for dense sands. The effects of a sustained static shear stress for dense sands was improved by modifying the plastic modulus for the effects that rotation of the back-stress ratio (represented by a factor $C_{K\alpha 1}$), such that it becomes,

$$K_p = G \cdot h_o \cdot \left[ \left( \alpha^p - \alpha \right):n \right]^{-0.5} \cdot \frac{C_{K\alpha}}{1 + C_{Kp} \left( \frac{z_{peak}}{z_{max}} \right) \left( \alpha^b - \alpha \right):n}$$

(72)

The parameter $C_{K\alpha}$ tracks the effect of rotation in the back-stress ratio as,

$$r_\alpha = \frac{\left( \alpha^p - \alpha \right):n + \left( \alpha^p - \alpha \right):n}{2 + \Delta_i}$$

(73)

$$C_{K\alpha} = 1 + C_\alpha \cdot \frac{\left( \alpha^p - \alpha \right):n - \left( \alpha^p - \alpha \right):n}{\left( \alpha^p - \alpha \right):n + \Delta_2}$$

(74)

The parameter $r_\alpha$ describes the degree of stress ratio rotation, with $r_\alpha = 0$ corresponding to a condition of full stress ratio reversal, and $r_\alpha \geq 1$ corresponding to a condition without any stress ratio reversal. In the expression for $C_{K\alpha 1}$, the term inside the MacCauley brackets is approximately equal to unity until the current back-stress ratio gets close to the previous back-stress ratio, at which point it quickly decreases to zero. In this manner, the $C_{K\alpha 1}$ term only causes an increase in plastic modulus for stress ratios that are between the initial back-stress ratio and previous back-stress ratio (i.e., a reloading condition). The parameters $\Delta_1$ and $\Delta_2$ are simply constants that preclude division by zero and/or allow a smooth transition as the other term in the denominator goes to zero; values of 0.01 and 0.05 for $\Delta_1$ and $\Delta_2$, respectively, were found to provide reasonable responses for the present model. The remaining parameter, $C_\alpha$, provides the magnitude of the increase in plastic modulus in the reloading condition, and can be used to increase the cyclic strength under conditions of a sustained static shear stress.
Effect of fabric on peak mobilized friction angles in drained and undrained loading

Kutter and Chen (1997) showed that plastic dilation rates are different in drained and undrained loading of the same clean sand, with the consequence being that the peak mobilized friction angles are also different for drained and undrained loading. This aspect of behavior would appear to be contradictory to having a bounding surface that is only dependent on the relative state of the sand (i.e., through the parameter $n^b$) if the mobilized friction angles for drained and undrained loading paths are both controlled by the bounding surface. The model proposed herein produces the same peak mobilized friction angles for drained and undrained loading because both conditions become limited by the same bounding surface. This aspect of behavior deserves closer examination in future efforts.

2.9 Summary of Constitutive Equations

The constitutive equations for the model presented herein are summarized in Table 2.1 along with the equations for the Dafalias-Manzari (2004) model.
Table 2.1. Comparison of constitutive equations

<table>
<thead>
<tr>
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<th></th>
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</thead>
<tbody>
<tr>
<td><strong>Critical state line</strong></td>
<td><strong>Critical state line</strong></td>
</tr>
<tr>
<td>$e_c = e_0 - \lambda \left( \frac{p_{cs}}{p_A} \right) \xi$</td>
<td>$\xi_R = \frac{R}{Q - \ln \left(100 \frac{p}{p_A} \right)} - D_R$</td>
</tr>
<tr>
<td><strong>Elastic deviatoric strain increment</strong></td>
<td><strong>Elastic deviatoric strain increment</strong></td>
</tr>
<tr>
<td>$d e^e = \frac{ds}{2G}$</td>
<td>$d e^e = \frac{ds}{2G}$</td>
</tr>
<tr>
<td>$G = G_A p_0 \frac{(2.97 - e)^2 \left( \frac{p}{p_A} \right)^{1/2}}{1 + e}$</td>
<td>$G = G_A p_0 \left( \frac{p}{p_A} \right)^{1/2} C_{SR} \left( 1 + \frac{z_{cum}}{4z_{max}} \right)$</td>
</tr>
<tr>
<td></td>
<td>$C_{SR} = 1 - C_{SR,0} \cdot \left( \frac{M}{M^b} \right)^{m_{SR}}$</td>
</tr>
<tr>
<td></td>
<td>$C_{SR,0} = 0.5$</td>
</tr>
<tr>
<td></td>
<td>$m_{SR} = 4$</td>
</tr>
<tr>
<td><strong>Elastic volumetric strain increment</strong></td>
<td><strong>Elastic volumetric strain increment</strong></td>
</tr>
<tr>
<td>$d e^v = \frac{dp}{K}$</td>
<td>$d e^v = \frac{dp}{K}$</td>
</tr>
<tr>
<td>$K = \frac{2(1+\nu)}{3(1-2\nu)} G$</td>
<td>$K = \frac{2(1+\nu)}{3(1-2\nu)} G$</td>
</tr>
<tr>
<td>Yield surface</td>
<td>Yield surface</td>
</tr>
<tr>
<td>---------------</td>
<td>---------------</td>
</tr>
<tr>
<td>$f = \left( (s - p\alpha) : (s - p\alpha) \right)^{\frac{1}{2}} - \sqrt{\frac{2}{3}}p m = 0$</td>
<td>$f = \left( (s - p\alpha) : (s - p\alpha) \right)^{\frac{1}{2}} - \sqrt{\frac{1}{2}}p m = 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Plastic deviatoric strain increment</th>
<th>Plastic deviatoric strain increment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d\varepsilon^p = \langle L \rangle R'$</td>
<td>$d\varepsilon^p = \langle L \rangle R'$</td>
</tr>
<tr>
<td>$R = B n - C \left( n^2 - \frac{1}{3} I \right) + \frac{1}{3} D I$</td>
<td>$R = R' + \frac{1}{3} D I = n + \frac{1}{3} D I$</td>
</tr>
<tr>
<td>$B = 1 + \frac{3}{2} \frac{1-c}{c} g(\theta, c) \cos(3\theta)$</td>
<td>$M^b = M \cdot \exp \left( -n^b \xi_R \right)$</td>
</tr>
<tr>
<td>$C = 3 \sqrt{\frac{3}{2} \frac{1-c}{c} g(\theta, c)}$</td>
<td>$M = 2 \cdot \sin(\phi_{cv})$</td>
</tr>
<tr>
<td>$c = \frac{Q_{ext}}{Q_{compr}}$</td>
<td>$\alpha^b = \sqrt{\frac{1}{2} \left[ M^b - m \right]} n$</td>
</tr>
<tr>
<td>$g(\theta, c) = \frac{2c}{(1 + c) - (1 - c) \cos(3\theta)}$</td>
<td></td>
</tr>
<tr>
<td>$M^b = M \cdot \exp \left( -n^b \xi \right)$</td>
<td></td>
</tr>
<tr>
<td>$\alpha^b = \sqrt{\frac{2}{3} \left[ g(\theta, c) M^b - m \right]} n$</td>
<td></td>
</tr>
<tr>
<td>$K_p = \frac{2}{3} G \cdot h_o \frac{(\alpha^b - \alpha) : n}{(\alpha - \alpha_m) : n}$</td>
<td>$K_p = G \cdot h_o \cdot \sqrt{\frac{\left[ (\alpha^b - \alpha) : n \right]^{0.5}}{\exp(\alpha - \alpha_m) : n - 1} + C_{\gamma^1} \cdot \frac{C_{\alpha}}{1 + C_{K_p} \left( \frac{Z_{\text{peak}}}{Z_{\text{max}}} \right) \left( (\alpha^b - \alpha) : n \right)}}$</td>
</tr>
<tr>
<td>$C_{\gamma^1} = \frac{h_o}{200}$</td>
<td>$C_{K_p} = 2$</td>
</tr>
<tr>
<td>Plastic volumetric strain increment</td>
<td>Plastic volumetric strain increment</td>
</tr>
<tr>
<td>------------------------------------</td>
<td>------------------------------------</td>
</tr>
<tr>
<td>( d\varepsilon_v^p = \langle L \rangle D )</td>
<td>( d\varepsilon_v^p = \langle L \rangle D )</td>
</tr>
<tr>
<td>( M^d = M \cdot \exp(n^d \xi) )</td>
<td>( M^d = M \cdot \exp(n^d \xi_R) )</td>
</tr>
<tr>
<td>( \alpha^d = \sqrt{2/3} \left[ g(\theta, c)M^d - m \right] n )</td>
<td>( \alpha^d = \sqrt{1/2} \left[ M^d - m \right] n )</td>
</tr>
<tr>
<td>( A_d = A_o (1 + \langle z : n \rangle) )</td>
<td>If dilating ((D &lt; 0)):</td>
</tr>
<tr>
<td>( D = A_d (\alpha^d - \alpha) : n )</td>
<td>( D = A_d \left( \langle \alpha^d - \alpha \rangle : n \right) )</td>
</tr>
</tbody>
</table>

If dilating \((D < 0)\):

\[
A_d = \frac{A_o (C_{z_{in2}})}{1 + \left( \frac{z_{cum}^2}{z_{max}} \right) \left( 1 - \frac{\langle -z : n \rangle}{\sqrt{2z_{peak}}} \right) (C_\varepsilon)(C_{pap})(C_{p\min})(C_{z_{in1}})}
\]
\[ A_{do} = \frac{1}{0.4} \left( \arcsin\left( \frac{M^b}{2} \right) - \arcsin\left( \frac{M}{2} \right) \right) \]

\[ C_{pzp} = \frac{1}{1 + \left( \frac{2.5p}{p_{zp}} \right)^5} \]

\[ C_{p_{\min}} = \frac{1}{1 + \left( \frac{p_{\min}}{p} \right)^5} \]

\[ C_{zin1} = 1.0 - \exp\left( -2.0 \frac{z_{ip} \cdot n - z \cdot n}{z_{max}} \right) \]

\[ C_{zin2} = \frac{1 + C_{zin1} \frac{z_{cum} - z_{peak}}{3z_{max}}}{1 + 3 \cdot C_{zin1} \frac{z_{cum} - z_{peak}}{3z_{max}}} \]

If contracting \((D \geq 0)\)

\[ D = A_{dc} \cdot \left[ (\alpha - \alpha^i) \cdot n + C_{in} \right]^2 \frac{(\alpha^d - \alpha) \cdot n}{(\alpha^d - \alpha) \cdot n + C_D} \]

\[ A_{dc} = \frac{A_{do} \left( 1 + \langle z \cdot n \rangle \right)}{\eta_p C_{dz}} \]

\[ C_{dz} = \left( 1 - \frac{z_{peak}}{z_{max}} \right) \left( \frac{z_{max}}{z_{max} + z_{cum}} \right) \geq \frac{1}{1 + \frac{z_{max}}{2}} \]

\[ C_D = 0.1 \]
Fabric-dilatancy tensor update

\[
dz = -c_z \left( -\frac{d\varepsilon^p_v}{2} \right) (z_{\text{max}} n + z)
\]

Fabric-dilatancy tensor update

\[
dz = -\frac{c_z}{1 + \left( \frac{z_{\text{cum}}}{2z_{\text{max}}} \right)} \left( -\frac{d\varepsilon^p_v}{2} \right) (z_{\text{max}} n + z)
\]

\[
dz_{\text{cum}} = |dz|
\]

Stress increment

\[
L = \frac{2Gn : de - n : rK\varepsilon_v}{K_p + 2G(B - C \cdot \text{tr} n^3) - KDn : r}
\]

\[
d\sigma = 2Gde + Kd\varepsilon_v I
\]

\[
- \langle L \rangle \left( 2G \left[ Bn - C \left( n^2 \frac{1}{3} I \right) \right] + KD I \right)
\]

Stress increment

\[
L = \frac{2Gn : de - n : rK\varepsilon_v}{K_p + 2G - KDn : r}
\]

\[
d\sigma = 2Gde + Kd\varepsilon_v I - \langle L \rangle (2Gn + KD I)
\]

\[
h_p = h_{po} \cdot \exp \left( -0.7 + 7.0 \left( 0.5 - \xi_R \right)^{2.5} \right) \quad \text{for} \quad \xi_R \leq 0.5
\]

\[
h_p = h_{po} \cdot \exp(-0.7) \quad \text{for} \quad \xi_R > 0.5
\]
Figure 2.1. Definition of the relative state parameter index, $\xi_R$ (Boulanger 2003a) and the effects of varying $Q$ and $R$. 
Figure 2.2. Schematic of yield, critical, dilatancy, and bounding lines in q-p space (after Dafalias & Manzari 2004).
Figure 2.3. Schematic of the bounding, dilation, and yield surfaces on the $r_{yy}$-$r_{xy}$ stress-ratio plane with the yield surface, normal tensor, dilatancy back stress ratio, and bounding back stress ratio.
Figure 2.4. Schematic showing the definitions of the initial back-stress ratio and previous initial back-stress ratio on the $\alpha_{yy}$-$\alpha_{xy}$ plane.
Figure 2.5. Undrained cyclic DSS loading response for $D_R = 35\%$ with an initial static shear stress ratio of $\alpha = 0.0$, showing the variation in stresses, stress ratios, and fabric tensor terms.
Figure 2.6. Undrained cyclic DSS loading response for $D_R = 35\%$ with an initial static shear stress ratio of $\alpha = 0.20$, showing the variation in stresses, stress ratios, and fabric tensor terms.
3. MODEL IMPLEMENTATION

The model has been implemented as a user defined material (udm) for use with the commercial finite difference program, FLAC 7.0 (Itasca 2011). This section includes a brief description of the mixed discretization scheme used in FLAC, the numerical implementation scheme used for PM4Sand, some additional comments on alternative implementation schemes, and information regarding the dynamic link library (DLL) for PM4Sand.

3.1 Aspects of FLAC's Numerical Approach

Explicit Integration

FLAC is an explicit finite difference program which uses time steps equal to or smaller than the minimum time required for waves to travel between any pair of nodes. This approach ensures that physical information does not propagate faster than numerical information. FLAC computes a default time step based on the properties of the model (e.g., element size, material stiffness, permeability, damping). Users may specify a time step that is smaller than the default value.

Obtaining numerically convergent solutions to nonlinear problems using FLAC requires that:

1) integration of the constitutive models be convergent, and
2) the explicit global solution is convergent.

The default time step computed by FLAC does not necessarily ensure a numerically convergent solution, especially for FLAC models that are subjected to very high loading rates. Convergence of the constitutive model's integration depends more strongly on the strain increment size, which is dependent on both the loading rate and time step size. Convergence of the explicit global solution depends more strongly on the sizes of the stress increments generated in the materials, which again are only indirectly controlled by the default time step size. For this reason, the user needs to evaluate the sensitivity of the solution to the time step size and not automatically assume that the default time step size ensures a convergent solution.

Mixed Discretization Scheme

FLAC uses a mixed discretization technique in which each quadrilateral zone (analogous to an element) is subdivided internally by its diagonals into two overlaid sets of constant-strain triangles. The term “mixed” stems from the fact that different discretizations are used for the isotropic and deviatoric parts of the strain and stress tensor (Marti and Cundall 1982). Isotropic stress and strain components are taken to be constant over the whole quadrilateral element, while the deviatoric components are treated separately for each triangular sub-element. Essentially, the shear strains are computed and maintained for each individual triangle, while the volumetric strains are computed for each quadrilateral as a weighted average of the volumetric strains within the juxtaposed pairs of triangles. Hour-glass modes are resisted by shear stresses generated in the triangles and the scheme accurately predicts plastic collapse loads because constant volume deformations are possible within the quadrilaterals. Note that discretization using triangles or four-node quadrilaterals alone would result in meshes that are over-constrained (too stiff) and which would tend to over-predict plastic...
collapse loads. Since each quadrilateral can be divided by two possible diagonals, a symmetric response of this discretization can only be obtained by running two complete meshes in parallel, each representing one half of the overall stiffness. At the end, the procedure reduces the number of constraints on plastic flow and, at the same time, reduces unwanted hourglassing by ensuring that hourglass modes produce non-zero stresses.

The implementation of a complex constitutive model in FLAC requires special attention to the way stresses and strains are handled under FLAC’s mixed discretization scheme. During each time step, FLAC calls the constitutive model once per triangular subzone (four times per zone). The isotropic components of the stress outputs from the four subzones are then averaged internally by FLAC according to the Mixed Discretization scheme. A consequence of this averaging of stresses is that the final stress state for any subzone is unlikely to satisfy the consistency condition of elasto-plastic models, meaning that the newly calculated stress states will not necessarily lie on the yield surface. Andrianopoulos (2006) addressed this problem by adopting a vanished elastic region in their elasto-plastic model.

3.2. Implementation of PM4Sand in FLAC

The implementation scheme for PM4Sand and how it relates to the challenges posed by FLAC’s mixed discretization scheme (section 3.1) are described here. Recall that each zone (consisting of four subzones) will start off at the beginning of each time step with a stress state and will be loaded by a strain increment whose volumetric components are the same in all four subzones while their deviatoric components are different (due to the mixed discretization scheme). The constitutive model will be called once per subzone (four times per zone) to obtain stresses from strains according to Equation 83 where $C_{ijkl}$ denotes the constitutive law:

\[
\sigma_i^{(i+1)} = \sigma_i^{(i)} + d\sigma^{(dt)}_i = \sigma_i^{(i)} + C_{ijkl} \varepsilon_{ij}^{(dt)}
\]

At the end of the step, each subzone will have its own stress state, which will be handled by FLAC independent of the constitutive model, and its own internal parameters, which FLAC will be unaware of. The subzones can therefore all have different stress states at the beginning of the next loading increment, and as such would need to maintain their own sets of internal parameters.

The current implementation scheme for PM4Sand is illustrated in Figure 3.1 and described by the pseudo-code listed in Table 3.1. At the end of each time step, the stress and internal variables are averaged over the four subzones. A drift correction is applied to ensure that the averaged stresses and internal variables satisfy the consistency condition; the correction involves projecting the back-stress ratio in the direction of the zone-averaged stress ratio. Another correction is applied if the zone-averaged stress ratio lies outside the bounding surface; the correction involves projecting the zone averaged stress ratio back along a normal to the bounding surface. The zone-averaged stresses are then used to compute a new dilatancy $D$ and plastic modulus $K_p$ that are consistent with the average response of the zone over this step. These values for $D$ and $K_p$ are then used by all four subzones in the next time step (i.e., the values of $D$ and $K_p$ lag one step behind the time step for which they were determined); note that this approach is used by other elasto-plastic models available in FLAC. Consequently, the four subzones will use a common $D$ and $K_p$ during each time step. Most other
internal parameters are also computed and retained at the zone level, as described by the pseudo-code in Table 3.1.

Two other implementation schemes for PM4Sand were explored for comparison purposes and found to have problems. The first of these alternative implementations was that used in version 1 of PM4Sand. In this implementation, each subzone had its own D and K_p and developed its own internal variables (e.g., fabric, back-stress-ratios, history terms) for each loading increment or step. At the end of the step, the stress and internal variables were averaged at the zone level and the drift and bounding surface corrections applied. The four subzones therefore started each loading step with a common set of stresses and internal variables, but each could have greatly differing values for D and K_p depending on the loading direction imposed on each subzone. In highly nonlinear loading steps, it was possible for one or two of the overlapping subzones to be strongly contractive (e.g., perhaps because of a reversal in loading direction) while the other subzones were strongly dilative, such that the incremental changes in stresses between the four subzones had competing effects on the zone's average behavior. This implementation was found to sometimes lead to unusual deformation modes in zones that were connected to piles by FLAC's interface springs. The unusual deformation modes are believed to be due to strong differences in loading directions and conditions between the subzones of zones being loaded by interface springs. This problem was effectively eliminated by the current implementation described in Table 3.1. The second of these alternative implementations increased the independence of the subzones, just to explore how it would affect behavior. In this implementation, each subzone (triangle) retained its own memory and history of stresses and internal parameters. This approach led to nonsensical results between the overlapping triangular subzones, especially when the loading conditions were highly nonlinear. For example, the external stresses sometimes could be carried by only two of the overlapping triangles (each having twice the correct stresses) while the stresses in the other two overlapping triangles went to zero. The experiences with these two alternative implementation schemes illustrate how FLAC's mixed discretization scheme requires special considerations when implementing highly nonlinear constitutive models.

Implementation of PM4Sand uses explicit integration and thus the user should routinely check that the solutions are not sensitive to time step size. The addition of substepping could improve the constitutive model's integration but would not eliminate the need to evaluate the effect of time step size on the global solution. In our experiences, the default time steps of FLAC in dynamic analyses of liquefaction problems have been small enough to ensure that numerical solutions are not significantly affected by time step size, and thus the additional computational cost of including substepping at the constitutive level was not considered necessary. Examples of the effects of time step size are presented in Section 3.3.

Numerical stability of the implemented model has been evaluated for a wide range of simulations of both element responses and system responses using the default range of parameters which are also summarized in the next section. Numerical stability problems may, however, develop when using input parameters which fall outside the ranges explored during model development, calibration, and implementation. Some initial bounds have therefore been placed on certain parameters whenever parametric analyses identified the potential for such problems; e.g., the minimum value of mean stress is limited to 0.5 kPa or 0.005 times the initial consolidation stress; the relative density was limited to values less than 1.2. The user must be aware that other limits may be identified as additional analyses explore a broader range of the possible input parameters.
3.3 Effect of Time Step Size on Element Responses

Numerical convergence of the current implementation of PM4Sand was evaluated by running numerous problems using a range of dynamic time steps (dydt), beginning with the default (maximum) time step computed by FLAC and then trying smaller and smaller values. These comparisons have shown that the solutions are not sensitive to the time step size for the range of problems and loading rates examined. The user must always check the sensitivity of boundary value problems to the time step size, however, as the accuracy of the explicit integration is strongly dependent on the size of the strain increments which are only partly controlled by the time step size.

For example, the effect of time step size (or strain increment size) on integration of the PM4Sand model is shown in Figure 3.3a for a single element simulation of a cyclic drained DSS test for sand at $D_R = 55\%$ at $\sigma'_vo = 100\text{kPa}$. The element was subjected to two cycles of strain-controlled loading with a single-amplitude shear strain of 1%. The strain rate was constant, with each cycle having a total duration of 1 sec (i.e., average loading frequency was 1 Hz). The default time step was 1.038e-4 s and the strain rate was 4 %/s which gives a step size of $\Delta \gamma = 4.15e-6 \%$/step. To evaluate different $\Delta \gamma$, the time step was reduced by factors of 1/2, 1/4, and 1/8. The simulated stress-strain responses showed minimal differences, indicating that the integration was sufficiently accurate for practical purposes.

A second example of the effect of time step size is shown in Figure 3.3b for a stress-controlled cyclic undrained DSS test for sand at $D_R = 55\%$ at $\sigma'_vo = 100\text{kPa}$. The default time step was 3.604e-5 s and the strain rate was again 4 %/s which gives a step size of $\Delta \gamma = 1.44e-6 \%$/step. The default time step is smaller for the undrained element test because of the higher wave speed in the pore water. To evaluate different $\Delta \gamma$, the time step was again reduced by factors of 1/2, 1/4, and 1/8. The simulated stress-strain responses again showed minimal differences, indicating that the integration was sufficiently accurate for practical purposes.

Figure 3.3a and b presents the same examples as Figure 3.2a and b but for a very high strain rate of 12%/s. The corresponding step sizes for the drained (a) and undrained (b) cases are 12.45e-6%/step and 4.32e-6 %/step, respectively. To evaluate different $\Delta \gamma$, the time step was again reduced by factors of 1/2, 1/4 and 1/8. The simulated stress-strain responses for the drained case showed minimal differences, whereas the undrained case showed some slight differences. The differences for the undrained case are attributed to the very high strain rate of 12%/sec, which was only used to examine the limits of behaviors.

Note that comparisons of solutions at different step sizes $\Delta \gamma$ cannot be made by varying the strain rate or cyclic loading frequency. FLAC is always solving the dynamic equation of equilibrium so changing the strain rate by changing any loading rate parameter also changes the dynamic excitation for the system, which can cause a change in the dynamic response of the element. In that case, any changes in the stress-strain response caused by changes in loading rate parameters may be a realistic simulation result that reflects the change in the dynamic excitation of the element.
3.4 Effect of Time Step Size on System Responses

The effect of time step size on the solution of full boundary value problems can similarly be examined by repeating simulations with successively smaller dynamic time step sizes. As an example, the effect of time step size on the response of a 2D simulation of a dynamic centrifuge model test (base simulation by Kamai 2011) is illustrated in Figure 3.4. Time histories of horizontal displacements at two points in the model are shown for the default time step size and for time step sizes that are 1/2, 1/4 and 1/8 of the default size. The differences in the horizontal displacements at the end of shaking are about 7.2%, 1.2% and 6.6%, respectively for Figure 3.3a and 2.2%, 0.1% and 4.8% for Figure 3.3b. The differences are illustrative of the range of effects observed in many cases, and are generally small enough for practical applications.

The sensitivity of simulation results to the dynamic time step size should always be evaluated as part of the sensitivity studies. As previously discussed in section 3.1, the effects of changing time step size may result from a combination of the effects on the constitutive model integration and the explicit global solution. Implementation of substepping in the constitutive model may reduce its effect, but will not remove the need to check the global solution's sensitivity to the step size. Since the sensitivity to step size should always be checked, the additional computational costs of including substepping at the constitutive level was not considered warranted at this time.

3.5 DLL Module

The PM4Sand model was coded in C++ and compiled as a DLL in Microsoft Visual Studio 2005. It has been tested in FLAC7 using the software’s option for User Defined Models (UDMs). The steps required for using a DLL are described in the FLAC manuals (Itasca 2011), and are thus only briefly summarized herein.

**Automatic loading of the DLL file**

1. Load the DLL file in the /Exe32/plugins/models subdirectory of the folder where FLAC has been installed.

2. Open the FLAC7.0 executable file or the FLAC graphical user interface. If the DLL is properly located, then the model should be automatically loaded. In order to verify that it has been loaded, the user can type “print model” in the console. If the model has been loaded then it should appear as “pm4sand” under the list of “Currently loaded CPP models”.

3. Before constitutive model plug-ins can be assigned to zones, the model must be configured for their use by giving the `config cppudm` command. Otherwise, the user will get a “model will not cycle” error message.

**Manual loading of the DLL file**

1. First, the user must make sure that the DLL file is located in the same folder together with the project file (*.prj) of the analysis.
In the project file (or the called fish file for the analysis) the model must be first configured for the use of constitutive model plug-ins (config cppudm) and then the model’s DLL can be loaded (model load modelpm4s003.dll). Again the user can verify the loading of the model by subsequently typing “print model” in the console.

In order to assign the model to the preferred zones the following command should be given:

```
model pm4sand i = ... j = ...
```
Figure 3.1 Schematic illustration of the averaging procedure followed in the implementation of PM4Sand: zone-averaged values are computed for some internal variables of the model, denoted as “m”, at the end of each step, after which other internal parameters, denoted as “q”, are computed based on the zone-averaged parameters.
Table 3.1: Simplified pseudo-code of PM4Sand (Version 2)

<table>
<thead>
<tr>
<th>Operations within one subzone:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Initialize the model parameters; this only happens when the model is first assigned or when FirstCall is set to zero at some point during the analysis. For detailed information on what parameters are initialized (or reset) see Table 3.2.</td>
</tr>
<tr>
<td>2. Obtain the strain increment from FLAC $d\varepsilon_\text{p}$.</td>
</tr>
<tr>
<td>3. Decompose the strain increment into volumetric and deviatoric components, $d\varepsilon_\text{p}$ and $d\varepsilon_\text{ij}$.</td>
</tr>
<tr>
<td>4. Calculate the trial elastic stress increment and trial elastic stress: $d\sigma_\text{tr} = \sigma_0^{ij} + d\sigma_\text{tr} = \sigma_0^{ij} + 2Gd\varepsilon_\text{ij} + Kd\varepsilon_\text{p}[I]$.</td>
</tr>
<tr>
<td>5. Calculate the trial stress ratio $r_{tr}^{ij}$, the distance from the yield surface $\text{dist}$, the unit normal to the yield surface $n^{ij}$ and the inner product of the change in back-stress ratio tensor with unit normal vector $daxn$.</td>
</tr>
<tr>
<td>$r_{tr}^{ij} = \frac{\sigma_{tr}^{ij} - p_{tr}[I]}{p_{tr}[I]}$</td>
</tr>
<tr>
<td>$\text{dist} = \sqrt{(r_{tr}^{ij} - \alpha_0^{ij}):(r_{tr}^{ij} - \alpha_0^{ij})}$</td>
</tr>
<tr>
<td>$n^{ij} = \frac{(r_{tr}^{ij} - \alpha_0^{ij})}{\text{dist}}$</td>
</tr>
<tr>
<td>$daxn = (\alpha_0^{ij} - \alpha_{in}^{ij}):(n^{ij})$</td>
</tr>
<tr>
<td>6. Check for yield:</td>
</tr>
<tr>
<td>a. If elastic then commit the trial stresses. Go to step 7. $\text{dist} &lt; \frac{1}{\sqrt{2}}m$ $\sigma_0^{ij} = \sigma_{tr}^{ij}$</td>
</tr>
<tr>
<td>b. If inelastic:</td>
</tr>
<tr>
<td>i. Calculate loading index L: $L = \frac{2Gn^{ij}:d\varepsilon_\text{p} - n:rd\varepsilon_\text{p}}{K_p + 2G - KD:n:r}$</td>
</tr>
<tr>
<td>ii. Calculate trial stress increment and trial stress: $d\sigma_\text{tr} = \sigma_0^{ij} + d\sigma_\text{tr} = \sigma_0^{ij} + 2Gd\varepsilon_\text{ij} + Kd\varepsilon_\text{p}[I] - L[2Gn^{ij} + KD[I]]$</td>
</tr>
<tr>
<td>iii. Apply penalties to stress ratios and back-stress ratios to meet the consistency condition and to remain within the bounding surface.</td>
</tr>
<tr>
<td>iv. Calculate image back-stress ratios and inner products: $\bar{\alpha}_0^{ij} = \frac{1}{\sqrt{2}}(M^b - m):n^{ij}$ $\bar{\alpha}_0^{ij} = \frac{1}{\sqrt{2}}(M^a - m):n^{ij}$</td>
</tr>
<tr>
<td>v. Commit the trial stresses (back-stress ratio, stress ratio, mean stress, stress)</td>
</tr>
<tr>
<td>7. Return all stress tensor components to FLAC (at this point FLAC takes over and will average them according to the mixed discretization scheme)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Operations referring to the whole zone:</th>
</tr>
</thead>
<tbody>
<tr>
<td>8. After the calculation has completed the 4th subzone, the following additional calculations are performed for the overall zone. Recall the following parameters for all 4 subzones and compute area-weighted average values for:</td>
</tr>
<tr>
<td>• Volumetric strain $d\varepsilon_\text{p}$</td>
</tr>
<tr>
<td>• Strain increment $d\varepsilon$</td>
</tr>
<tr>
<td>• Mean stress $\bar{p}$</td>
</tr>
<tr>
<td>• Stress tensor (committed one) $\bar{\sigma}_0^{ij}$</td>
</tr>
<tr>
<td>• Back-stress ratio tensor $\bar{\alpha}_0^{ij}$</td>
</tr>
<tr>
<td>• Unit normal to yield surface vector $\bar{n}^{ij}$</td>
</tr>
</tbody>
</table>
9. Apply penalties to the averaged zone parameters to meet the consistency condition and remain within the bounding surface.
10. Calculate image back-stress ratios and inner products for the averaged zone parameters.
11. Track reversals in the stress-ratio based on averaged parameters.
12. Compute Dilatancy D and Plastic Modulus K_p for the past average step in the zone.
14. If dilating \((D<0)\), update the fabric tensor for the zone and if exceeding its former value, update the cumulating fabric term.
\[
\mathbf{z}^{ij} = \mathbf{z}^{ij} - \frac{c_z}{\max\left(1, \frac{\overline{\varepsilon}_p}{\overline{\varepsilon}_{\text{cum}}}, \frac{1}{2}\frac{\mathbf{z}_{\text{max}}}{\mathbf{z}_{\text{max}}}\right)} \frac{\overline{\varepsilon}_p}{D} (\mathbf{z}_{\text{max}} \mathbf{n}^{ij} + \mathbf{z}^{ij})
\]
15. Update the relative state parameter, the bounding and dilation stress ratios, the elastic shear modulus (depends on fabric) and the elastic bulk modulus for the next step.
16. Update the initial and previous initial back-stress values and the strain increment accumulators.
17. Update initial back-stress ratios upon reversal.
18. Commit zone stress tensor, zone mean stress, zone back-stress ratio tensor, zone stress-ratio tensor to memory.
Table 3.2: Initialization function of PM4Sand (called during the first application of the model and whenever FirstCall=0)

1. Set thresholds for mean effective stress:
   \[ p_{\text{min}} = \frac{p_{\text{atm}}}{200} \quad \text{and} \quad p_{\text{min}2} = 10p_{\text{min}} \]

2. Calculate critical state stress ratio:
   \[ M = 2\sin(\varphi_{cv}) \]

3. Obtain stresses from FLAC and create stress tensor (these will be the committed stresses from which the calculation will start):
   \[ \sigma^i_0 \]

4. Check stresses and calculate mean effective stress:
   a. If stresses compressive (following FLAC’s sign convention that tensile stresses and strains are positive):
      \[ \sigma^i_o < 0 \to p_o = \min\left(p_{\text{min}}, \frac{1}{2} \sigma^i_o \right) \]
      \[ p_{\text{min}} = \min\left(p_{\text{min}}, \frac{p}{200} \right) \]
      \[ p_{\text{min}2} = \min\left(p_{\text{min}2}, \frac{p}{20} \right) \]
   b. If stresses tensile:
      \[ \sigma^i_o > 0 \to p_o = -\frac{p_{\text{atm}}}{20} \]
      \[ \sigma^i_o = p_o \cdot [l] \]

5. Calculate relative state parameter and subsequently calculate the bounding and dilation stress ratios and \( A_{do} \) (from input property \( D_R \) and secondary parameters \( R, Q, n^b \) and \( n^d \) – see Chapter 4):
   \[ \xi_R = \frac{R}{Q - \ln\left(-\frac{100p_{\text{atm}}}{p_{\text{atm}}}\right)} - D_R \]
   a. If dense-of-critical (\( \xi_R < 0 \)):
      \[ M^b = Mexp\left(-n^b\xi_R\right), \quad M^d = Mexp\left(n^d\xi_R\right) \]
      \[ A_{do} = 2.5 \left[ \frac{\sin^{-1}\left(\frac{M^b}{2} - \varphi_{cv}\right)}{M^b - M^d} \right] \]
   b. If loose-of-critical (\( \xi_R > 0 \)):
      \[ M^b = Mexp\left(-\frac{n^b}{4}\xi_R\right), \quad M^d = Mexp\left(4n^d\xi_R\right), \quad A_{do} = 1.24 \]

6. Check that initial stresses are inside the bounding surface (or dilation surface if it is greater) and compute the committed back-stress and stress ratio tensors from the stress tensor:
   \[ M^{\text{cut}} = \max(M^b, M^d), \quad M^{\text{fin}} = -2 \cdot \sqrt{\frac{1}{2} \left( \sigma^i_o - p_o [l] \right) \left( \sigma^i_o - p_o [l] \right)} \]
   a. If \( M^{\text{fin}} > M^{\text{cut}} \):
      \[ r^i_o = \frac{\sigma^i_o - p_o [l]}{p_o} \left( \frac{M^{\text{cut}}}{M^{\text{fin}}} \right) \]
      \[ \sigma^i_o = p_o [l] + r^i_o p_o \]
      \[ \alpha^i_o = r^i_o \frac{M^{\text{cut}} - m}{M^{\text{cut}}} \]
   b. If \( M^{\text{cut}} > M^{\text{fin}} \):
7. Create/Initialize the initial back-stress ratio, initial previous back-stress ratio, minimum initial back-stress ratio and maximum initial back-stress ratio tensors (see also Section 2.5 on Stress Reversal):

\[ r^{ij}_{o} = \left( \frac{\sigma^{ij}_{o} - p_{o}[l]}{p_{o}} \right) \]

\[ \alpha^{ij}_{o} = r^{ij}_{o} \]

\[ \alpha^{ij}_{in} = \alpha^{ij}_{o} \]

\[ \alpha^{ij}_{inm} = \alpha^{ij}_{inmax} = \alpha^{ij}_{inmin} = \alpha^{ij}_{in} \]

8. Calculate initial values of elastic shear modulus, elastic bulk modulus, plastic modulus, dilatancy:

\[ G = G_{o}p_{atm} \frac{-p_{o}}{p_{atm}} \]

\[ K = G \frac{2(1 + \nu)}{3(1 - 2\nu)} \]

\[ K_{p} = 100G \]

\[ D = 0 \]

9. Initialize fabric related terms (see Section 2.8) – note that these terms will be referring to the whole zone:

\[ p_{xp} = \frac{p_{o}}{100} \]

\[ z_{peak} = \frac{z_{max}}{100000} \]

\[ z_{xp} = z_{:p} = 0 \]

\[ z_{xp}p_{k} = -z_{max} \frac{p_{o}}{50} \]

\[ z^{ij}_{:e} = z^{ij}_{:in} = z^{ij}_{:u} = z_{cum} = 0 \]
Figure 3.2 Effect of dynamic time step on the results obtained from (a) drained and (b) undrained cyclic DSS element test simulations ($D_R=55\%$, $\sigma'_{vo}=1$ atm) loaded at a shear strain rate of $12\%/s$. The black line in each case denotes the response obtained with FLAC’s default dynamic time step.
Figure 3.3 Effect of dynamic time step on the results obtained from (a) drained and (b) undrained cyclic DSS element test simulations ($D_R=55\%$, $\sigma'_v=1$ atm) loaded at a shear strain rate of 12%/s. The black line in each case denotes the response obtained with FLAC’s default dynamic time step.
Figure 3.4 Comparison of the effect of the variation of the dynamic time step on the results obtained from the analysis of a 2D mesh (SSK01 – Kamai 2011). Results are presented at two points within the mesh.
4. MODEL INPUT PARAMETERS AND RESPONSES

4.1 Model input parameters

The model parameters are grouped into two categories; a primary set of five parameters (three properties, one flag, and atmospheric pressure) that are most important for model calibration, and a secondary set of parameters that may be modified from their default values in special circumstances.

Primary input parameters

The three primary input parameters are the sand’s apparent relative density \( D_R \), the shear modulus coefficient \( G_o \), and the contraction rate parameter \( h_{p0} \). These three parameters are discussed below and summarized in Table 4.1.

Relative density (\( D_R \)) can be estimated in practice by correlation to penetration resistances. For example, a common form for SPT correlations is,

\[
D_R = \frac{(N_1)_{60}}{C_d}
\]

(76)

where \( D_R \) is expressed as a ratio rather than a percentage. Idriss and Boulanger (2008) reviewed published data and past relationships, and then adopted a value of \( C_d = 46 \) in the development of their liquefaction triggering correlations. For the CPT, they similarly reviewed available relationships and arrived at the following expression,

\[
D_R = 0.465 \left( \frac{q_{clN}}{C_{dq}} \right)^{0.264} - 1.063
\]

(77)

for which they adopted \( C_{dq} = 0.9 \). For the example loading responses shown later, \( D_R \) values of 35%, 55%, and 75% were used, which would correlate to SPT \((N_1)_{60}\) values of 6, 14, and 26 by the above correlations.

The input value of \( D_R \) is best considered an "apparent relative density," rather than a strict measure of relative density following conventional laboratory tests. The input value of \( D_R \) influences the response of the model and thus it is just another input parameter that the user can adjust as part of the calibration process. The above correlations are provided for the purpose of obtaining a reasonable estimate for the apparent \( D_R \) so that the resulting model behaviors are also reasonable. There are situations, however, where the user may choose to adjust the input \( D_R \), up or down relative to the above relationships, to improve its calibration to some other relationship or data.

The second primary input parameter is the constant \( G_o \) which will control the elastic (or small strain) shear modulus as,
The elastic shear modulus can be calibrated to fit in-situ $V_s$ measurements, according to,

$$G = G_o p_A \left( \frac{p}{p_A} \right)^{1/2}$$  \hspace{1cm} (78)

or alternatively fit to values of $V_s$ that may be estimated by correlation to penetration resistances. For the examples shown herein, the correlation by Andrus and Stokoe (2000) in Figure 4.1 was used, with a slight modification that constrains the extrapolation to very small $(N_1)_{60}$ values, as shown in the figure,

$$V_{sf} = 85 \left[ (N_1)_{60} + 2.5 \right]^{0.25}$$  \hspace{1cm} (80)

The above relationships in combination with the default values for the maximum and minimum void ratios (described later) produce $V_{s1}$ values of 144, 171, and 196 m/s and corresponding $G_o$ values of 468, 677, and 907 for the $D_R$ values of 35%, 55%, and 75%, respectively.

Alternatively, the above expressions were combined together with a range of typical densities to arrive at the following simpler expression for estimating $G_o$,

$$G_o = 167 \sqrt{(N_1)_{60} + 2.5}$$  \hspace{1cm} (81)

This expression produces $G_o$ values of 476, 677, and 890 for the $D_R$ values of 35%, 55%, and 75%, respectively.

The third primary input parameter is the constant $h_{po}$ which is used to modify the contractiveness and hence enable calibration of the model to specific values of cyclic resistance ratio (CRR). For the examples presented herein, the target CRR values were based on the liquefaction triggering correlation by Idriss and Boulanger (2008) in Figure 1.2. This relationship produces target CRR values for an effective overburden stress of 1 atm and an earthquake magnitude of M=7.5 of 0.090, 0.147, and 0.312 for the corresponding SPT $(N_1)_{60}$ values of 6, 14, and 26, respectively. The corresponding values of $h_{po}$ to achieve these CRR are 0.96, 0.71 and 0.98, respectively.

The value of atmospheric pressure, $p_A$, should also be specified in the unit set being used for the analysis. If not specified, it will default to 101,300 Pascal.

The flag FirstCall is used to re-set the back-stress ratio history terms equal to the current stress ratio, and to erase all fabric terms. The first time the model is called, the flag should be unspecified or have a value of 0. The model will then initiate the back-stress ratios and all pertinent history terms using the current state of stress. The flag is then set equal to 1.0 internally. If FirstCall is later set equal to 0.0 using the property command in FLAC, this will cause the material to re-initiate all
internal terms, thereby re-setting the back-stress and stress ratio history terms and erasing all fabric terms. FirstCall should usually be set to 0.0 just before initiating dynamic earthquake loading. Otherwise, the model will retain memory of the loading during the static initiation of the model, which may or may not be desired.
Table 4.1 – Primary input parameters (parameter names in square brackets correspond to the input name to be used within FLAC)

<table>
<thead>
<tr>
<th>Parameter [FLAC name]</th>
<th>Comments</th>
</tr>
</thead>
</table>
| $D_R$ [$D_r$]         | **Apparent relative density:** Primary variable controlling dilatancy and stress-strain response characteristics. Input as a fraction, not as a percentage. Commonly estimated based on CPT or SPT penetration resistances, such as the following relationships used by Idriss and Boulanger (2008):

$$D_R = \sqrt{\frac{(N_r)_{60}}{C_d}}$$

with $C_d = 46$, and

$$D_R = 0.465 \left( \frac{q_{c1N}}{C_{d_0}} \right)^{0.264} - 1.063$$

with $C_{d_0} = 0.9$. |
| $G_o$ [$G_o$]         | **Shear modulus coefficient:** Primary variable controlling the small strain shear modulus, $G_{\text{max}}$. Should be chosen to match estimated or measured shear wave velocities according to $G_{\text{max}} = \rho V_s^2$. A default value for $G_o$ is computed based on the specified $D_R$, using a relationship derived from typical densities and the modified correlation between SPT $(N_1)_{60}$ and $V_{s1}$ values shown in Figure 4.1. First, an equivalent SPT $(N_1)_{60}$ is computed as,

$$(N_1)_{60} = C_d \left( D_R \right)^2$$

using $C_d = 46$, and then the value of $G_o$ is computed as,

$$G_o = 167 \sqrt{(N_1)_{60} + 2.5}$$ |
| $h_{po}$ [$h_po$]     | **Contraction rate parameter:** Primary variable that adjusts contraction rates and hence can be adjusted to obtain a target cyclic resistance ratio, as commonly estimated based on CPT or SPT penetration resistances and liquefaction correlations. |
| $p_A$ [$P_{\text{atm}}$] | **Atmospheric pressure** in the unit set being used. Defaults to 101,300 Pascals if not specified. |
| FirstCall [First_Call] | **Flag** used to re-set the back-stress ratio history terms equal to the current stress ratio, and to erase all fabric terms. First_Call should usually be set to 0.0 at model initiation and/or just before initiating dynamic earthquake loading. Otherwise, the model will retain memory of the loading during the static initiation of the model, which may or may not be desired. |
Secondary input parameters

Secondary input parameters are those parameters for which default values have been developed that will generally produce reasonable agreement with the trends in typical design correlations. The user must, however, still confirm through element loading calibrations that the default parameters are appropriate for their particular conditions. The secondary input parameters (16 in total) are listed in Table 3.2, along with commentary on the recommended default values. The selected values for these parameters have been embedded within the initialization section of the code and unless specified otherwise by the user, they are applied by default.

The parameters $C_z$, $C_\alpha$, and $C_\varepsilon$ cannot be set to zero directly, but rather must be given either a very small value or a negative value (in which case they are assigned very small values internally). If these parameters are given values of zero, then the code cannot distinguish them from values that have not been initialized, and hence will assign them their default values.

Table 4.2 – Secondary input parameters

<table>
<thead>
<tr>
<th>Parameter [FLAC name]</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_o$ [h_o]</td>
<td>Variable that adjusts the ratio of plastic modulus to elastic modulus. The default value of $h_o=(0.25+D_R)/2$, with a minimum value of 0.30, was chosen to provide reasonable $G/G_{max}$ and damping relationships for the default value of $G_o$. This variable may require adjustment in combination with any adjustments to $G_o$.</td>
</tr>
<tr>
<td>$e_{max}$ and $e_{min}$ [e_max] [e_min]</td>
<td>The maximum and minimum void ratios affect the computation of density, and affect how volumetric strains translate into changes in relative state. Default values of 0.8 and 0.5, respectively, were adopted. Refinements in these parameters for a practical problem may not be necessary, as the calibration of other parameters will have a stronger effect on monotonic or cyclic strengths.</td>
</tr>
<tr>
<td>$n^b$ [n_b]</td>
<td>Default value is 0.50. Controls dilatancy and thus also the peak effective friction angles. Note that $M^b$ for loose of critical states is computed using $n^b/4$.</td>
</tr>
<tr>
<td>$n^d$ [n_d]</td>
<td>Default value is 0.10. Controls the stress-ratio at which contraction transitions to dilation, which is often referred to as phase transformation. A value of 0.10 produces a phase transformation angle slightly smaller than $\phi_{cv}$, which is consistent with experimental data. Note that $M^d$ for loose of critical states is computed using $4n^d$.</td>
</tr>
<tr>
<td>$A_{do}$ [A_do]</td>
<td>Default value is computed based on Bolton's dilatancy relationship at the time of initialization; typical values will be between 1.2 and 1.5.</td>
</tr>
<tr>
<td>Parameter</td>
<td>Default Value and Details</td>
</tr>
<tr>
<td>-----------</td>
<td>--------------------------</td>
</tr>
</tbody>
</table>
| **$z_{\text{max}}$**  
**[$z_{\text{max}}$]** | Default value is 0.28 $\exp(6.1D_R)$, with an upper limit of 40. This returns values of 2.4, 8.0, and 27.2 at $D_R$ of 35, 55, and 75%, respectively. May require varying if the relationship between $D_R$ and cyclic strength is significantly different from that implied by the liquefaction correlation of Idriss and Boulanger (2008) – i.e., if the value of $h_{\infty}$ is substantially different from the range of values used in these calibration examples. |
| **$c_z$**  
**[$c_z$]** | Default value is 250. Controls strain levels at which fabric affects become important. To set this parameter to zero, the user must input a negative value. |
| **$c_e$**  
**[$c_e$]** | Default value varies with $D_R$. The value is 5.0 for $D_R$ less than 35%, and linearly decreases to its minimum value of 1.0 at $D_R = 75%$. Can be used to adjust the rate of strain accumulation in undrained cyclic loading. To set this parameter to zero, the user must input a negative value. |
| **$\phi_{cv}$**  
**[$\phi_{cv}$]** | Default value is 33 degrees. |
| **$\nu_o$**  
**[$\nu_o$]** | Default value is 0.30. For 1-D consolidation of an elastic material, the value of $K_0$ would correspond to,  
$$K_0 = \frac{\nu}{1 - \nu}$$  
The default value for $\nu$ results in a $K_0$ value of 0.43 in 1-D consolidation. |
| **$C_{\text{GD}}$**  
**[$C_{\text{GD}}$]** | Default value is 2.0. The small-strain elastic modulus degrades with increasing cumulative plastic deviator strains ($z_{\text{cum}}$). The maximum degradation approaches a factor of $1/C_{\text{GD}}$. |
| **$C_\alpha$**  
**[$C_\alpha$]** | Default value varies with $D_R$. The value is 0.0 for $D_R$ less than 55%, and linearly increases to its maximum value of 8.0 at $D_R = 75%$. This variable controls the effect that sustained static shear stresses have on plastic modulus. To set this parameter to zero, the user must input a negative value. |
| **$Q$**  
**[$Q$]** | Default value is 10. Default value is for quartzitic sands per recommendations of Bolton (1986). |
| **$R$**  
**[$R$]** | Default value is 1.5. Default value for quartzitic sands would be 1.0 per recommendations of Bolton (1986); a slight increase in $R$ is used to lower the critical state line to better approximate typical results for direct simple shear loading. |
| **$m$**  
**[$m$]** | Default value is 0.01. Default value provides reasonable modeling and numerical stability. |

**Tracking variables**

Many of the parameters internal to PM4Sand may be tracked for debugging purposes. The table below lists six internal parameters that may be of interest. Other internal parameters that can be tracked include: $pzp$, $zxp$, $pmin$, $pmin2$, $Caln$, $MM$, $alfa_{11}$, $alfa_{12}$, $max_G$, $max_K$, $r_{11}$, $r_{12}$, $aln_{11}$, $aln_{12}$, $alnP_{11}$, $alnP_{12}$, $z_{11}$, $z_{12}$, $zcum$, $Cka1$, $Cka2$, $zpeak$, $zxpPk$, $eqsum$, $evsum$, $LoadInd$, $Dilat$, $Kp$, $zabs$, $evol$, $eq_{11}$, $eq_{22}$, $eq_{12}$, $epsIncr$ and $daxn$. 

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Table 4.3 – Internal parameters available for tracking

<table>
<thead>
<tr>
<th>Parameter [FLAC Name]</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M^b$ [$Mb$]</td>
<td>Bounding surface stress ratio</td>
</tr>
<tr>
<td>$M^d$ [$Md$]</td>
<td>Dilation surface stress ratio</td>
</tr>
<tr>
<td>$M_{cur}$ [$M_{cur}$]</td>
<td>Current stress ratio</td>
</tr>
<tr>
<td>$G$ [$shearG$]</td>
<td>Elastic shear modulus</td>
</tr>
<tr>
<td>$K$ [$bulkK$]</td>
<td>Elastic bulk modulus</td>
</tr>
<tr>
<td>$\tilde{\xi}_R$ [$rsp$]</td>
<td>Relative state parameter</td>
</tr>
<tr>
<td>$D_R$ [$Dr$]</td>
<td>Relative density, which evolves in response to volumetric strains. Note that the input parameter $D_r$ is an initial parameter and does not evolve during an analysis.</td>
</tr>
</tbody>
</table>

Recall that internal parameters (properties) can be accessed using the `z_prop` command of FLAC. For example, an algorithm to find the maximum bulk modulus in a model can be:

```plaintext
loop $i (1, izones)
  loop $j (1, jzones)
    $dummy = z_prop($i,$j,'bulkK')
    $maxK  = max($dummy,$maxK)
  end_loop
end_loop
```
4.2 Example calibration and model responses for a range of element loading conditions

The response of the model is illustrated in this section by presenting simulation results for a set of input parameters that were calibrated to emphasize realistic modeling of liquefaction behavior. Results are presented for sands having initial apparent relative densities of 35%, 55%, and 75% with corresponding SPT \((N_1)_{60}\) values of approximately 6, 14, and 26, respectively, based on the correlations presented previously. All secondary input parameters were assigned the default values summarized previously in Table 4.2. Values for \(G_0\) were obtained using the previously presented correlation between SPT \((N_1)_{60}\) values and overburden-corrected shear wave velocity \(V_{S1}\) (Figure 4.1). Values for \(h_{po}\) were obtained by matching the CRR values from direct simple shear (DSS) simulations with the CRR\(_{M=7.5}\) values that were computed using the SPT-based liquefaction triggering correlation by Idriss and Boulanger (2006, 2008); an SPT-based estimate of CRR for an \(M=7.5\) earthquake and effective overburden stress of 1 atm was assumed to be approximately equal to the CRR corresponding to 15 uniform loading cycles causing a peak shear strain of 3% in direct simple shear loading. Back-stress ratio and fabric history terms were re-initialized, using FirstCall=0, prior to the start of loading; This step was only important for the cyclic undrained loading cases where the element was first consolidated under an applied horizontal shear stress. The model input parameters for the examples presented in this section are summarized in Table 4.4 below.

Table 4.4. Input parameters for example element responses

<table>
<thead>
<tr>
<th>Scenario field condition</th>
<th>Model input parameters (^{(a)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_r) (\text{(N}<em>1\text{)}</em>{60}) (V_{S1}) using Andrus &amp; Stokoe (2000) (\text{CRR}<em>{M=7.5}) using Idriss &amp; Boulanger (2008) (D_r) (G_0) (h</em>{po})</td>
<td></td>
</tr>
<tr>
<td>0.35 6 144 0.090</td>
<td>0.35 476 0.96</td>
</tr>
<tr>
<td>0.55 14 171 0.147</td>
<td>0.55 677 0.71</td>
</tr>
<tr>
<td>0.75 26 196 0.312</td>
<td>0.75 890 0.98</td>
</tr>
</tbody>
</table>

\(^{(a)}\) All secondary input parameters were assigned the default values listed in Table 4.2.

Undrained cyclic loading

The undrained cyclic loading responses for the calibrated models are illustrated in Figures 4.2-4.4. These figures show the stress-strain and stress-path responses for undrained uniform cyclic loading in DSS with a vertical consolidation stress of 1 atm and initial static shear stress ratios \((\alpha)\) of 0.0, 0.1, and 0.2. Results for initial \(D_r\) of 35%, 55%, and 75% are presented in Figures 4.2, 4.3, and 4.4, respectively. Close up views of the stress-strain responses for \(D_r=35\%, 55\%, \) and 75% with an initial static shear stress ratio \(\alpha = 0.0\) are presented in Figure 4.5.

The stress-strain responses for \(\alpha = 0.0\) illustrate the model's ability to progressively reach larger and larger shear strains with continued cyclic loading, rather than locking up in a repeating loop as many plasticity models do. The ability to simulate the progressive accumulation of shear strains
reflects the inclusion of the cumulative fabric terms, as described previously. The progressive increases in peak shear strain after the soil has reached a peak excess pore pressure ratio ($\tau_a$) greater than 98% are realistic in magnitude.

The stress-strain responses with nonzero initial static shear stresses show a progressive accumulation of shear strains in the direction of the initial static shear stress, with the rate and nature of the stress-strain response also being reasonably realistic.

**CRR versus number of loading cycles – Effect of $D_R$ and failure criterion**

The CRR obtained for the calibrated models are summarized in Figure 4.6 showing the cyclic stress ratio (CSR) required to cause an excess pore pressure ratio ($\tau_a$) of 98% or single-amplitude shear strains of 1% and 3% versus number of uniform loading cycles. These results are for DSS loading with a vertical consolidation stress of 1 atm, an initial $K_o$ of 0.5, and zero initial static shear stress ratio ($\alpha=0.0$). The simulation results in this figure were fitted with a power law, for which the exponent "b" is labeled beside each curve.

The slopes of these CRR versus number of loading cycles are in good agreement with typical values obtained in laboratory testing studies. The exponent b is generally between 0.24 and 0.27 for these simulations. For the experimental data in Figure 1., the exponent b ranges from a low of 0.1 for one study on to a high of 0.34 for another. The ability of the model to produce reasonable slopes for these curves is attributed primarily to the changes in the plastic modulus and dilatancy relationships (Ziotopoulou and Boulanger 2012).

The slopes of the CRR versus number of loading cycle curves can be slightly adjusted by the parameters $n^b$ and $n^d$. For example, repeating the same simulations for $D_R=75\%$ with $n^b=0.8$ (versus the value of 0.5 used herein) and with all other factors the same, increases the exponent b to values of 0.28 to 0.33. Ziotopoulou and Boulanger (2012) performed the same simulations using an earlier set of calibration parameters and got values for the exponent b ranging from 0.26 to 0.36. Note, however, that a greater value for $n^b$ also results in greater peak friction angles and changes other responses as well, so such adjustments cannot be made independent of other features of behavior.

**CRR versus number of loading cycles – Effect of confining stress**

The effect of overburden stress on CRR for the calibrated models is illustrated in Figure 4.7 showing the CSR required to cause a single-amplitude shear strain of 3% versus number of uniform loading cycles for different confining stresses. These results are for DSS loading with initial $K_o=0.5$, initial static shear stress ratio ($\alpha$) of 0.0, and vertical consolidation stresses of 1, 4, and 8 atm. The cyclic strengths for $D_R=35\%$ are the least affected by confining stress, while the cyclic strengths for $D_R=75\%$ are the most affected (reduced).

The equivalent $K_{\sigma}$ values from these simulations, with the CRR values compared at 15 uniform loading cycles, are compared in Figure 4.8 to the relationships recommended by Boulanger and Idriss (2004) based on the framework presented in Boulanger (2003b). The simulated effects of confining stress are in good agreement, as expected since the expression for $h_{ps}$ was calibrated to this relationship.
Summary plots of the CSR required to cause a single-amplitude shear strain of 3% versus number of uniform loading cycles are presented in Figure 4.9 for different values of initial static shear stress ratio. Results are presented for sand at DR=35%, 55%, and 75% loaded in DSS with an initial K_o=0.5, a vertical consolidation stress of 1 atm, and with initial static shear stress ratios (α) of 0.0, 0.1, 0.2, and 0.3. The simulation results are reasonable in predicting that the presence of an initial static shear stress ratio results in lower cyclic strengths for loose sands (e.g., the DR = 35% results) and greater cyclic strengths for denser sands (e.g., the DR=75% results). The simulations are also reasonable in that they show that this effect is actually dependent on the confining stress (as built into the model by its dependence on relative state). However, the simulations for this calibration cause rotations in the curves relating CRR to the number of uniform loading cycles for some conditions. These types of responses have not been observed in experimental data, and thus this is an attribute of the model that warrants further work.

The performance of the model simulations is further evaluated in Figure 4.10 by comparing the same simulations for DR = 75% (run with the default value of C_α) against results obtained using C_α = 0.0, which effectively turns off the model terms that were added to account for the effects of shear stress and fabric rotation. The simulations with C_α = 0.0 cause the CRR to decrease significantly with increasing value of initial static shear stress ratio, which is contrary to experimental data for sands at this range of relative states. Note that the parameter C_α has no effect on the results for DR = 35% because its relative state (under the vertical stress of 1 atm) is below the reference level at which stress ratio and fabric rotation effects are nonzero, and it has only a small effect on the results for DR = 55%. Thus, the current model formulation improves the ability to approximate the effects of initial static shear stress ratios, despite the limitations discussed above.

Summary plots of the CSR required to cause a single-amplitude shear strain of 3% versus number of uniform loading cycles are presented in Figure 4.11 for different values of the lateral earth pressure coefficient at rest K_o (i.e., the ratio of horizontal to vertical effective stresses at the time of consolidation). Results are presented for sand at DR=35%, 55%, and 75% loaded in DSS, a vertical consolidation stress of 1 atm, and with zero initial static shear stress ratio.

The response for drained monotonic loading in direct simple shear (DSS) and plane-strain compression (PSC) for sand at DR of 35%, 55%, and 75% under vertical confining stresses of ¼, 1, 4, and 16 atm is shown in Figures 4.12 and 4.13. The responses reasonably approximate the effects of relative density and confining stress on both the stress-strain and volumetric strain responses. The plots show the response up to shear strains of 10%, while the simulations tend to reach critical state conditions at shear strains of 40-60%. The post-peak rate of strain-softening is dictated by the dilation rate, which is constrained to approximate Bolton's (1986) stress-dilatancy relationship. The simulated post-peak softening is slower than often observed in experimental results (e.g., Figure 1.9) because drained laboratory experiments are often affected by strain localizations in dilating sands.
The rate of strain-softening in a dilating zone is much lower than represented by global measurements of stress and strain.

The peak effective friction angles from simulations of drained monotonic loading in DSS and plane strain compression (PSC) are shown versus vertical consolidation stress in Figure 4.14 where they are also compared to Bolton’s (1986) relationship for plane strain conditions for Q=10 and R=1.5 (which are the default values of Q and R that have been selected for the model). The peak friction angles are lower in DSS than in PSC because of the difference in how the friction angles are computed for this plot. For PSC, the peak friction angle is computed based on the peak stress ratio within the element, without any predetermined assumptions regarding the orientation of the plane on which it will occur. For DSS loading, the peak friction angle was computed as the inverse tangent of the peak stress ratio on the horizontal plane, following the same convention commonly used in practice for interpreting such tests. In the DSS simulation, however, the horizontal plane was not the plane of maximum stress obliquity, and therefore the interpreted peak friction angle is slightly lower than the value obtained in PSC. Computationally, both the DSS and PSC mobilize similar peak friction angles if the comparison is made only for the plane of maximum stress obliquity in both simulations; In the DSS, however, the stress ratio on the horizontal plane in the DSS simulations is often closer to \( \sin(\phi) \) as opposed to \( \tan(\phi) \), which results in the apparent differences shown in Figure 4.14. Despite these differences, the peak friction angles are reasonable and consistent with typical design correlations (e.g., Kulhawy and Mayne 1990).

**Undrained monotonic loading**

The undrained monotonic loading in direct simple shear (DSS) for sand at \( D_R \) of 35%, 55%, and 75% under vertical consolidation stresses of \( \frac{1}{4}, 1, 4, \) and 16 atm are shown in Figure 4.15, while the same responses are shown with normalization by the vertical consolidation stress in Figure 3.16. The stress-strain responses show strain hardening behavior at lower relative states than would be expected based on the experimental results for reconstituted sands, such as presented by Yoshimine et al. (1999). Experiments on loose reconstituted sands often show strain softening to some minimum shear stress ratio (e.g., quasi-steady state condition) before beginning to strain harden, and that minimum stress ratio is often in the range of 0.1 to 0.3. For the present calibration, the CRR for \( D_R = 35\% \) sands under 1 atm of confining stress was targeted to be 0.090 based on a field-based liquefaction correlation, and it was not possible to calibrate the model to match both the target CRR values and the monotonic undrained strengths presented in Yoshimine et al. (1999). If the monotonic behavior was more important than the CRR values, then a different calibration would be required.

**Drained, strain-controlled, cyclic loading from small to large strains**

Drained strain-controlled cyclic loading in DSS for sand at \( D_R \) of 35%, 55%, and 75% under vertical consolidation stresses of 1, 4, and 16 atm with \( K_o=1.0 \) are shown in Figures 4.17 to 4.19, with results also shown for the equivalent modulus reduction (\( G/G_{max} \)) and equivalent damping ratio versus cyclic shear strain amplitude. Also shown on these figures are the modulus reduction and equivalent damping ratio curves recommended for sands at different depths by EPRI (1993). The simulated modulus reduction and equivalent damping ratio curves depend on the effective confining stress in a pattern and magnitude that is consistent with empirical design correlations, such as the ones by EPRI. The simulated modulus reduction curves for this calibration tend to be slightly higher than the empirical curves, whereas the simulated equivalent damping ratios are in reasonable
agreement with the empirical curves over a fairly broad range of shear strain amplitudes. The model response for this calibration avoids the problem common to many plasticity models of producing excessively high equivalent damping ratios as shear strain amplitudes approach about one percent.

**Drained, strain-controlled cyclic loading: Densification under large numbers of cycles**

Drained strain-controlled cyclic loading in DSS for sand at Dₚ of 35%, 55%, and 75% subjected to 10 cycles at 1% shear strain under a vertical effective stress of 1 atm are presented in Figure 4.20 to illustrate the accumulation of volumetric strains with increasing number of constant-amplitude strain cycles. The model response with this calibration produces volumetric strains that are about twice the values expected based on the empirical data presented in Figure 1.12, although the general pattern of stress-strain behavior and its dependency on Dₚ and confining stress are reasonably consistent with the empirical data. Alternative model calibrations can produce better agreement with these behaviors, but they were generally found to require compromising the fit to the CRR correlations.

**Post-liquefaction reconsolidation strains**

The volumetric strains that develop during post-liquefaction reconsolidation of sand are difficult to model using the conventional separation of strains into elastic and plastic components because a large portion of the post-liquefaction reconsolidation strains are due to sedimentation effects (i.e., volume reductions while the effective stresses remain close to zero) which are not easily incorporated into either the elastic or plastic components of behavior. For example, it is common for many plasticity-based constitutive models to predict reconsolidation volumetric strains from a condition of rᵤ=100% that are only a fraction of one percent, whereas experimental data show values ranging from one to four percent for most relative densities (e.g., Figure 1.12).

Volumetric strains due to post-cyclic reconsolidation are plotted in Figure 4.21 versus the maximum shear strain induced during undrained cyclic loading. Results are shown for sand at Dₚ = 35%, 55%, and 75% loaded in DSS with an initial Kₒ=0.5, a vertical consolidation stress of 1 atm, and zero initial static shear stress ratio. After cyclic loading to different maximum shear strains, the horizontal shear stress was reduced to zero such that the excess pore pressure was near its maximum possible value (e.g., rᵤ was approximately 98% or larger for cases with maximum shear strains of 3% or greater). The computed volumetric strains were less than about 0.25%, which are much smaller than expected based on common experimental data.

The fact that most elastic-plastic constitutive models do not adequately model post-liquefaction reconsolidation strains can be important for some practical problems. In the first version of this model (Boulanger 2010), an approximate method for including the effects of sedimentation strains was included. This approach was later found to lead to poor element behaviors in some boundary value problems and thus was omitted from this version.
Figure 4.1. Correlation between overburden-corrected shear wave velocity and SPT penetration resistances in clean sands (after Andrus and Stokoe 2000).
Figure 4.2. Undrained cyclic DSS loading response for $D_R = 35\%$ with vertical effective consolidation stress of 1 atm and with initial static shear stress ratios of 0.0, 0.1, and 0.2.
Figure 4.3. Undrained cyclic DSS loading response for $D_R = 55\%$ with vertical effective consolidation stress of 1 atm and with initial static shear stress ratios of 0.0, 0.1, and 0.2.
Figure 4.4. Undrained cyclic DSS loading response for $D_R = 75\%$ with vertical effective consolidation stress of 1 atm and with initial static shear stress ratios of 0.0, 0.1, and 0.2.
Figure 4.5. Undrained cyclic DSS loading responses for $D_R = 35\%, 55\%, \text{ and } 75\%$ with a vertical effective consolidation stress of 1 atm and without any initial static shear stress.
Figure 4.6. Cyclic stress ratios versus number of equivalent uniform loading cycles in undrained DSS loading to cause ru=98% or single-amplitude shear strains of 1% or 3% for DR = 35, 55, and 75% with a vertical effective consolidation stress of 1 atm. Each set of CSR-N simulations was fit with a power relationship and the exponent b labeled beside each curve.
Figure 4.7. Cyclic stress ratios versus number of equivalent uniform loading cycles in undrained DSS loading to cause single-amplitude shear strain of 3% for $D_R = 35, 55, \text{ and } 75\%$ with vertical effective consolidation stresses of 1, 4, and 8 atm.
Figure 4.8. Comparison of $K_\sigma$ factors, determined at 15 uniform loading cycles to cause 3% single-amplitude shear strain, from simulations versus relationships recommended by Boulanger and Idriss (2004).
Figure 4.9. Cyclic stress ratios versus number of equivalent uniform loading cycles in undrained DSS loading to cause single-amplitude shear strain of 3% for $D_R = 35, 55,$ and 75% with vertical effective consolidation stresses of 1 atm and initial static shear stress ratios of 0.0, 0.1, 0.2, and 0.3.
Figure 4.10. Effect of stress ratio and fabric rotation terms on the relationship between cyclic stress ratio and number of equivalent uniform loading cycles to cause single-amplitude shear strain of 3% in undrained DSS loading for $D_R = 75\%$ with vertical effective consolidation stress of 1 atm and initial static shear stress ratios of 0.0, 0.1, 0.2, and 0.3: Top graph is for the default value of $C_\alpha = 8.0$, and bottom graph is for $C_\alpha = 0.0$ (set through inputting a negative value).
Figure 4.11. Cyclic stress ratios versus number of equivalent uniform loading cycles in undrained DSS loading to cause single-amplitude shear strain of 3% for $D_R = 35$, 55, and 75% with vertical effective consolidation stresses of 1 atm and initial $K_o$ values of 0.3, 0.5, 0.8, and 1.2.
Figure 4.12. Drained monotonic DSS loading responses for $D_R = 35\%, 55\%$, and $75\%$ with vertical effective confining stresses of $1/4, 1, 4, 16$, and $64$ atm and $K_o=0.5$. 
Figure 4.13. Drained monotonic PSC (plane strain compression) loading responses for $D_R = 35\%$, $55\%$, and $75\%$ with initial isotropic confining stresses of $\frac{1}{4}$, 1, 4, 16, and 64 atm.
Figure 4.14. Peak friction angles from drained monotonic PSC and DSS loading responses for $D_R = 35$, 55, and 75% under effective confining stresses of $\frac{1}{4}$, 1, 4, 16, and 64 atm. For DSS loading, the friction angle is presented using the conventional interpretations that the horizontal plane is the failure plane (the actual plane of peak stress ratio is not horizontal in these simulations).
Figure 4.15. Undrained monotonic DSS loading responses for $D_R = 35\%$, $55\%$, and $75\%$ under vertical effective consolidation stresses of $\frac{1}{4}$, 1, 4, and 16 atm.
Figure 4.16. Normalized responses to undrained monotonic DSS loading for \( D_R = 35, 55, \) and 75\% under vertical effective consolidation stresses of \( \frac{1}{4}, 1, 4, \) and 16 atm.
Figure 4.17. Drained strain-controlled cyclic DSS loading responses for $D_R = 35\%$ under vertical effective consolidation stresses of 1, 4, and 16 atm.
Figure 4.18. Drained strain-controlled cyclic DSS loading responses for $D_R = 55\%$ under vertical effective consolidation stresses of 1, 4, and 16 atm.
Figure 4.19. Drained strain-controlled cyclic DSS loading responses for \( D_R = 75\% \) under vertical effective consolidation stresses of 1, 4, and 16 atm.
Figure 4.20. Volumetric strains during drained strain-controlled cyclic DSS loading for $D_R = 35$, $55$, and $75\%$ with a vertical effective consolidation stress of 1 atm.
Figure 4.21. Volumetric strain due to post-cyclic reconsolidation versus the maximum shear strain induced during undrained cyclic DSS loading.
5. CONCLUDING REMARKS

The PM4Sand (version 2) plasticity model presented herein is built upon the basic framework of the stress-ratio controlled, critical state compatible, bounding surface plasticity model for sand presented by Dafalias and Manzari (2004). A series of modifications and additions to the model were incorporated by Boulanger (2010; version 1) and further herein (version 2) to improve its ability to approximate the stress-strain responses important to geotechnical earthquake engineering practice; in essence, the model was calibrated at the equation level to provide for better approximation of the trends observed in empirical correlations commonly used in practice. These constitutive modifications included:

- revising the fabric formation/destruction to depend on plastic shear rather than plastic volumetric strains;
- adding fabric history and cumulative fabric formation terms;
- modifying the plastic modulus relationship and making it dependent on fabric;
- modifying the dilatancy relationships to provide more distinct control of volumetric contraction versus expansion behavior;
- providing a constraint on the dilatancy during volumetric expansion so that it is consistent with Bolton’s (1986) dilatancy relationship;
- modifying the elastic modulus relationship to include dependence on stress ratio and fabric history;
- modifying the logic for tracking previous initial back-stress ratios (i.e., loading history effect);
- recasting the critical state framework to be in terms of a relative state parameter index;
- simplifying the formulation by restraining it to plane strain without Lode angle dependency for the bounding and dilation surfaces; and
- providing default values for all but three primary input parameters.

The three primary model input parameters are: the shear modulus coefficient, $G_0$, which should be calibrated to the estimated or measured in-situ shear wave velocity; the contraction rate parameter, $h_{po}$ which is used to calibrate to the estimated in-situ cyclic resistance ratio; and an apparent $D_R$ which affects the peak drained and undrained strengths and the rate of strain accumulation during cyclic loading.

The model (version 2) was implemented as a user defined material in a dynamic link library for use with the commercial program FLAC (Itasca 2011). The implementation includes an improved numerical procedure for working with the mixed discretization scheme of FLAC.

The behavior of the model was illustrated by simulations of element loading tests covering a broad range of conditions, including drained and undrained, cyclic and monotonic loading under a range of initial relative densities, confining stresses, and initial shear stress conditions. The simulations presented in this report were completed using the dynamic link library (DLL) version modelpm4s003.dll compiled on May 15, 2012. The model is shown to provide reasonable approximations of desired behaviors and to be relatively easy to calibrate.
ACKNOWLEDGMENTS

The authors are indebted to Professors Bruce Kutter and Yannis Dafalias for their discussions and insights regarding the formulation and implementation of constitutive models. Robbie Jaeger was instrumental to the development of the dynamic link library. The beta versions of the model were extensively utilized by Ronnie Kamai and Jack Montgomery, whose feedback resulted in improvements to the model and this manual. The comments and results from trial applications by Lelio Mejia, Erik Newman, and Richie Armstrong were extremely helpful and are also greatly appreciated.

REFERENCES


