The Chow ring of hyperkähler manifolds

Arnaud Beauville

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The Chow ring

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$$CH^p(X) = \{n_1 Z_1 + \ldots + n_k Z_k\}/\sim \quad CH_p := CH^{n-p}$$

$Z_i$ irreducible of codimension $p$, $\sim =$ rational equivalence (generalizes linear equivalence of divisors).
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Unlike cohomology, the Chow ring is poorly understood.

It is usually very large: if $X$ has a nontrivial holomorphic form, $CH_0(X)$ cannot be parametrized by an algebraic variety (Roitman).
The Chow ring of K3 surfaces

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1. All points of $S$ lying on a rational (singular) curve have the same class $c_S$ in $CH_0(S)$.
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3. $c_2(S) = 24c_S$. 

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**Proof of 1 and 2**: easy consequence of:

**Theorem (Mumford-Bogomolov, Mori-Mukai)**

Any curve \( C \subset S \) is linearly equivalent to a sum of rational curves.
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Any curve \( C \subset S \) is linearly equivalent to a sum of rational curves.

(Intuitive reason: by Riemann-Roch, \( \dim |C| = g(C) \).)
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$R \rightarrow H \text{ ample} \rightarrow R'$

$p \rightarrow p'$
Proof of ① and ②

\[ R \quad \xrightarrow{H'} \quad R' \]

\[ p \quad \xrightarrow{} \quad p' \]
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\[ \Rightarrow \quad [p] = [p'] \text{ in } CH_0(S). \]

Proof of ②: \( C \cdot C' \sim \sum C \cdot R_i \sim \sum x_{ij} \) with \( x_{ij} \in R_i \).
is much more involved. We deduce it from the vanishing of the modified diagonal cycle in $CH_2(S \times S \times S)$ (choosing some $r \in R$): 

$$\{(x, x, x)\} - \{(r, x, x)\} + \text{permutations} + \{(r, r, x)\} + \text{permutations}$$
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Remarks

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Theorem 1 is quite particular to K3 surfaces: O’Grady has examples of \( S_d \subset \mathbb{P}^3 \) with \( \text{rk}(\text{Im } \mu) \geq \left[ \frac{d - 1}{3} \right] \).
A reformulation
Consider the graded ideal $CH^\bullet_{hom}(X)$ of $CH^\bullet(X)$:

$$0 \to CH^p_{hom}(X) \to CH^p(X) \to H^{2p}(X, \mathbb{Z})$$
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$\leadsto$ one-step filtration of $CH(X)$: $F^0 = CH(X), F^1 = CH(X)_{hom}$. 
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\(1\) and \(2\) $\iff$ **Multiplicative splitting** of this filtration:
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$$CH = CH_{(0)} \oplus CH_{\text{hom}}$$, $CH_{(0)}$ stable under multiplication.
Consider the graded ideal $CH^*_\text{hom}(X)$ of $CH^*(X)$:

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For $S$ K3:

$$CH^1(S) = \text{Pic}(S) \oplus (0)$$

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**Question**: For which other varieties do we have such a splitting?
Abelian varieties
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A abelian variety: natural splitting

\[ \text{Pic}(A) \otimes \mathbb{Q} = \text{Pic}^+(A) \oplus \text{Pic}^-(A) \text{ of } (\pm 1)\text{-eigenspaces for } (-1_A)^*, \]
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**Convention**: From now on, $CH$ means $CH \otimes \mathbb{Q}$. 
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**Theorem (O’Sullivan, 2011)**

\[ \exists \text{ multiplicative splitting } \quad CH(A) = CH(A)_{(0)} \oplus CH(A)_{\text{hom}}, \]  
extending the previous one for \( CH^1 \).
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extending the previous one for \(CH^1\).

\(CH(A)_{(0)}\) is the space of ”symmetrically distinguished cycles”. The construction is quite involved (80 pages).
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*(WSP)* Let $DCH(X)$ be the subalgebra of $CH(X)$ spanned by divisors. The cycle class map $DCH(X) \to H(X, \mathbb{Q})$ is injective.

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Or equivalently:

Any polynomial relation $P(D_1, \ldots, D_k) = 0$ between divisor classes in $H(X, \mathbb{Q})$ already holds in $CH(X)$. 

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Voisin has refined (WSP) to incorporate part 3 of Theorem 1:

(WSP$^+$) The cycle class map is injective on the subalgebra of $CH(X)$ spanned by divisors and the Chern classes of $X$. 

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For which varieties does \((WSP)\) or \((WSP^+)\) hold?
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For which varieties does (WSP) or (WSP$^+$) hold?

**Claim:** (WSP) does **not** hold for all Calabi-Yau varieties.

**Lemma**

$X \to Y$ surjective, (WSP) for $X$ $\implies$ (WSP) for $Y$. 
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**Lemma**

\[ \text{X } \to \text{ Y surjective, (WSP) for X } \Rightarrow \text{ (WSP) for Y.} \]

**Proof**:

\[
\begin{align*}
DCH(X) & \hookrightarrow H(X, \mathbb{Q}) \\
DCH(Y) & \rightarrow H(Y, \mathbb{Q}).
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**Example:** \( b : Y \to \mathbb{P}^3 \) blow up of \( C \subset \mathbb{P}^3 \) of genus 2, degree 5; \( E \) exceptional divisor. Then \( \text{Pic}(Y) = \langle b^* H, E \rangle \). For general \( C \), \( b^* H^2, b^* H \cdot E, E^2 \) linearly independent in \( DCH^2(Y) \),
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Then $X := \text{double covering of } Y \text{ branched along } D \in | -2K_Y |$ is a Calabi-Yau threefold, $DCH^2(X) \leftrightarrow H^4(X)$.  

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Then $X :=$ double covering of $Y$ branched along $D \in \mid -2K_Y \mid$ is a Calabi-Yau threefold, $DCH^2(X) \leftrightarrow H^4(X)$.

However, for a Calabi-Yau **hypersurface** $X$ of dimension $n$:

$$CH^p(X) \otimes CH^{n-p}(X) \xrightarrow{\mu} \mathbb{Q} \cdot h^n \subset CH^n(X) \quad (1)$$
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**Question**: Is there a larger (natural) class of Calabi-Yau manifolds for which (1) holds?
Conjecture

\((WSP^+)\) holds for projective hyperkähler manifolds.
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Here hyperkähler = irreducible holomorphic symplectic (IHS) = simply-connected + $H^0(X, \Omega^2_X) = \mathbb{C}\sigma$, $\sigma$ symplectic 2-form.
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Recall: Many interesting properties, but very few examples.
Hyperkähler manifolds

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\textbf{Recall :} Many interesting properties, but very few examples.

\textbf{Up to deformation,} only two series in each (even) dimension:

1. for \(S\ \text{K3}, \ S^{[n]} := \text{Hilbert scheme} = \{Z \subset S \mid \text{length}(Z) = n\}\)

\(\text{= desingularization of the symmetric product } \text{Sym}^n S.\)
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2. $K_n$ ("generalized Kummer varieties"): analogous construction starting from $S = \text{abelian surface}$.
Hyperkähler manifolds

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+ 2 sporadic examples in dimension 6 and 10 (O’Grady).
Deformations

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Recall: For each $g$, one 19-dimensional moduli space $\mathcal{F}_g$ of K3 surfaces $S \subset \mathbb{P}^g$ ($S_4 \subset \mathbb{P}^3$, $S_{2,3} \subset \mathbb{P}^4$, etc.)
Recall: For each \( g \), one 19-dimensional moduli space \( \mathcal{F}_g \) of K3 surfaces \( S \subset \mathbb{P}^g \) (\( S_4 \subset \mathbb{P}^3 \), \( S_{2,3} \subset \mathbb{P}^4 \), etc.)

The \( S^{[n]} \) for \( S \in \mathcal{F}_g \) form only a \textit{hypersurface} in the deformation space of \( S^{[n]} \), which has dimension 20.
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We say that \( X \) is of type \( K3^{[n]} \) if it is deformation equivalent to \( S^{[n]} \) for some K3 surface \( S \); same for type \( K_n \).
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We say that $X$ is of type $K3^{[n]}$ if it is deformation equivalent to $S^{[n]}$ for some K3 surface $S$; same for type $K_n$.

Challenge: Describe explicitely complete families of projective varieties of type $K3^{[n]}$. 
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Other examples: O’Grady, Iliev-Ranestad, Debarre-Voisin ($n = 2$); Iliev-Kapustka$^2$-Ranestad ($n = 3$), Lehn$^2$-Sorger-v. Straten ($n = 4$).
No example known for type $K_n$. 
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**Proposition (Voisin)**

Let $S$ be a K3 surface, $\tau := \text{rk } H^2(S)_{tr} = 22 - \text{rk } \text{Pic}(S)$. Then $(\text{WSP}^+)$ holds for $S^{[n]}$ for $n \leq 2\tau + 4$, in particular for $n \leq 8$. 
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**Idea**: Using de Cataldo-Migliorini, reduce to analogous statement for $S^n$: for $n \leq 2\tau + 1$, $DDCH(S^n) \hookrightarrow H(S^n)$, where $DDCH(S^n) := \text{subalgebra of } CH(S^n) \text{ spanned by pull back of divisors in } S \text{ and the diagonal in } S \times S$. 

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The Chow ring of hyperkähler manifolds
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Then write down complete list of relations between these generators of $\text{DDCH}(S^n)$, and check that they hold already in $\text{CH}(S^n)$.
Remark (Q. Yin): Can we go one step further, namely prove

\[ DDCH(S^n) \hookrightarrow H(S^n) \] for \( n = 2\tau + 2 \)?
**Remark** (Q. Yin):

\[ \text{DDCH}(S^n) \hookrightarrow H(S^n) \text{ for } n = 2\tau + 2 \iff " \wedge^{\tau+1} H^2(S)_{\text{tr}} = 0" \]

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So (WSP) for $S^{[n]}$ implies nothing for type $K3^{[n]}$. 
Riess’ theorem
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**Easy part:** $\text{Ker} \left( S \cdot \text{Pic}(X) \otimes \mathbb{C} \to H^\bullet(X, \mathbb{C}) \right) = \text{ideal spanned by classes } D^{n+1} \text{ for } D \in \text{Pic}(X) \otimes \mathbb{C}, \ q(D) = 0 \ (\text{Bogomolov}).$
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Thus (WSP) $\iff$ for these classes, $D^{n+1} = 0$ in $CH(X) \otimes \mathbb{C}$. 

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The Chow ring of hyperkähler manifolds
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Riess’ theorem: hard part

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The Chow ring of hyperkähler manifolds
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$f : X \xrightarrow{\varphi} X' \xrightarrow{p} \mathbb{P}^n$, $X'$ HK, $\varphi$ birational, $p$ Lagrangian fibration.
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$p^*H^{n+1} = 0$ in $CH(X') \Rightarrow D^{n+1} = \varphi^*p^*H^{n+1} = 0$ in $CH(X)$. □
The Bloch-Beilinson filtration
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**Conjecture (Bloch-Beilinson)**

For every $X$ smooth projective, $\exists$ filtration $F^\bullet$ on $CH(X)$:

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which is functorial (both for $f^*$ and $f_*$) and multiplicative.
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**Hope:** For hyperkähler manifolds, the B-B filtration admits a multiplicative splitting, i.e. comes from a graded ring structure:

$$CH^p(X) = CH^p_{(0)} \oplus \ldots \oplus CH^p_{(i)} \oplus \ldots \oplus CH^p_{(p)} \quad \underbrace{\oplus \ldots \oplus}_{F^i}$$
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Recent work of Voisin gives some evidence in the case of $\text{CH}^0$:

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The Chow ring of hyperkähler manifolds
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For $x \in X$, put $O_x := \{y \in X \mid y \sim_{\text{rat}} x\}$. 

Arnaud Beauville  The Chow ring of hyperkähler manifolds
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For any projective $X$, $\text{gr}^p_{F^*} CH_0(X)$ should be controlled by $H^0(X, \Omega^p_X)$; thus for $X$ HK of dimension $2n$, $F^{2p-1} = F^{2p}$ and

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For $x \in X$, put $O_x := \{ y \in X \mid y \sim_{\text{rat}} x\}$. 

$O_x$ is a countable union of closed subvarieties $Z$ which are isotropic – i.e. $\sigma|_Z = 0$. In particular $\dim O_x \leq n$. 

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The Chow ring of hyperkähler manifolds
The conjectural splitting

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**Example**: For $S$ K3, $S^1(S) = \{ x \in S \mid [x] = c_S \text{ in } CH_0(S) \}$, $S^1 CH_0(S) = \mathbb{Q} \cdot c_S$. 
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**Conjecture** (Voisin): The filtration $F^\bullet$ and $S^\bullet$ are opposite; i.e., if $CH_{(j)} := S^{n-j} \cap F^{2j}$:

$$CH_0(X) = CH_{(0)} \oplus \ldots \oplus CH_{(2i)} \oplus \ldots \oplus CH_{(2j)} \oplus \ldots \oplus CH_{(2n)}.$$

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The Chow ring of hyperkähler manifolds
Some evidence (Voisin)
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$$\dim S^i(X) = 2n - i \implies CH_0(X) = S^{n-i} + F^{2i+2}.$$
Ingredients of the proof
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**Proposition**

\[ Z \subset S^i(X) \text{ irreducible of dimension } 2n - i \Rightarrow Z \text{ coisotropic} \]

\((T^\perp_Z \subset T_Z) \text{ and } \exists f : Z \to B, \text{ fibers of } f = \text{orbits.}\)
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By expected properties of B-B filtration,

\[ \Rightarrow S^{n-i}CH_0(X) \rightarrow CH_0(X)/F^{2i+2}. \]
THE END
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Happy birthday, Ron!