First Semester

1. Optimal Taxation in an Endogenous Growth Model (45 pts) Consider an economy with two forms of capital, physical and human. There is a large number of ex-ante identical households with CRRA utility and common discount factor $\beta$ who accumulate these two forms of capital over time. Firms operate homogenous production functions which exhibits constant returns to scale in these two factors, so that:

$$Y_t = F(K_t, H_t)$$

Each form of capital accumulates via direct use of the final good via a law of motion with depreciation factors $\delta_k$ and $\delta_h$, so that for each household:

$$k_{t+1} = (1 - \delta_k)k_t + x_t$$

$$h_{t+1} = (1 - \delta_h)h_t + e_t$$

Firms rent each form of capital in competitive markets at rates $\hat{r}_t$ and $\hat{w}_t$. There is no uncertainty.

1. Define a competitive equilibrium for this economy and characterize it via necessary and sufficient conditions on allocations and prices assuming all choices are interior.

2. Give a condition for there to be positive long-run growth in the economy.

Now suppose that the government must fund a fixed sequence of spending $(g_t)_{t=0}^\infty$ using only proportional taxes on each source of income. The tax rates need not be equal.

3. Define a tax-distorted competitive equilibrium. How does the condition for positive long-run growth change if the tax rates are constant?

4. Derive the implementability condition(s) for a TDCE of this economy.

5. If the government chooses taxes to maximize household welfare subject to the restriction derived above then what happens to these tax rates as $t \to \infty$. What about $t > 0$?

2. Asset Pricing: Houses and Rent (45 pts) Consider an economy with two types of agents, each making up half of the population. One household type is a “renter” and one is an “owner”. Households receive endowments of the consumption good equal to $e_{i,t}$ each
period, with \( i \in \{r, o\} \). The owner-type's endowment is constant and normalized to one, whereas the renter-type's endowment is stochastic and follows a two-state Markov process (all renters have the same endowment).

Denoting the amount of consumption a household enjoys in a given period as \( c_t \) and the amount of housing services as \( h_t \), a period's utility is given by:

\[
    u(c_t, h_t) = \frac{c_t^{1-\sigma_c}}{1-\sigma_c} + \theta \frac{h_t^{1-\sigma_h}}{1-\sigma_h}
\]

Houses are long-lived real assets in fixed supply (use the parameter \( H \) for the net supply). Each house generates one unit of housing services per period, perfectly divisible. Each period, the owner of a house (or fraction thereof) can either use it for services or rent those services in exchange for consumption goods (and of course he can rent additional housing services if he wishes). At the end of each period, an owner-type household can then sell his share of houses and buy some more for the next period. Renter-type households cannot own houses directly, they can only rent services.

1. Define a Arrow-Debreu equilibrium for this economy in which households have access to a full set of set of state contingent securities. Define a history.
2. Describe the dynamics of consumption and housing services for each household type, i.e. how do the allocations for each household depend on the history up to \( t \)?
3. Solve for the unique stable house price as a function of the state of the economy under the assumption that \( \sigma_c = \sigma_h = \sigma \). Hint - this involves a simple linear system if you make a change of variables on the house price equation.
4. People often look at the price-to-rent ratio in a city to determine if it is better to buy or rent, with the advice for potential home buyers that a low value means to buy and a high value means to rent. Can this model justify that advice when \( \sigma_c = \sigma_h \)?
2 Second Semester

1. Optimal monetary policy under commitment (20 points)
   Consider a model of optimal monetary policy when the central bank can commit to a state-contingent path for its instrument.
   
   The central bank's objective is to minimize the loss function
   \[ E_t \sum_{j=0}^{\infty} \beta^j \left[ \pi_{t+j}^2 + \lambda x_{t+j}^2 \right] \]
   subject to the constraint
   \[ \pi_t = (1 - \phi) \beta E_t \pi_{t+1} + \phi \pi_{t-1} + \kappa x_t + \epsilon_t \]
   where the central bank's instrument is \( x_t \) and \( \beta, \lambda, \phi, \) and \( \kappa \) are model parameters. Moreover, \( \epsilon_t \) is an AR(1) shock that follows
   \[ \epsilon_t = \rho \epsilon_{t-1} + \nu_t \]
   where \( \rho \in (0, 1) \) and \( \nu_t \) is an iid, mean zero innovation.

   Assume that the central bank can commit at date 0 to a contingent path for \( x_t \). Argue rigorously how the parameter \( \phi \) determines the following aspects of the solution of the model under commitment:
   a) Forward-looking vs. history-dependent nature of optimal policy.
   b) Time-consistency of optimal policy.

   For concreteness in your arguments, you can consider three different values of \( \phi : 0, 1, \) and an intermediate value of \( \phi \in (0, 1) \).
2. Flexible price model with monopolistic competition (20 points)
Consider a monopolistic competition model with preference shocks and flexible prices.
The representative consumer's problem is

$$\max \ E_t \sum_{j=0}^{\infty} \psi_{t+j} \beta^j \left[ \frac{C_{t+j}^{1-\sigma}}{1-\sigma} - \frac{N_{t+j}^{1+\phi}}{1+\phi} \right]$$

subject to the flow budget constraint

$$P_tC_t + B_t = (1 + i_{t-1})B_{t-1} + W_tN_t + \Gamma_t$$

where $C_t$ is consumption of the composite good, $N_t$ is hours supplied, $P_t$ is the welfare-relevant price level of the composite final good, $W_t$ is nominal wage, $B_t$ is period $t$ holdings of one-period risk-less nominal bonds in zero net supply that yield a gross return $(1 + i_t)$ in period $t + 1$, and $\Gamma_t$ is aggregate profits from firms that are owned by the consumer.

$\psi_t$ is a preference shock that follows an exogenous stationary process.

Finally, $\beta$ is the discount factor, $\sigma$ is the inverse of the intertemporal elasticity of substitution, and $\phi$ is the inverse of the Frisch elasticity of labor supply.

Assume that the agent is subject to a no-Ponzi game constraint. Moreover, the consumer takes the profit flow from the firms as given. The labor market is competitive.

The composite good $C_t$ is a Constant Elasticity of Substitution aggregate of a continuum of varieties $c_t(i)$, indexed by $i$

$$C_t = \left[ \int_0^1 c_t(i)^{\frac{1}{1-\xi}} \, di \right]^{\frac{1}{1-\xi}}$$

where $\xi > 1$ is the elasticity of substitution among the varieties.

a) First, consider the expenditure minimization problem of the consumer

$$\min \ \int_0^1 p_t(i)c_t(i) \, di$$

subject to

$$\left[ \int_0^1 c_t(i)^{\frac{1}{1-\xi}} \, di \right]^{\frac{1}{1-\xi}} = C_t$$

where $p_t(i)$ is the price of variety $i$. Derive the FOC with respect to $c_t(i)$. Use the FOC to show that the optimal price index $P_t$ is given by

$$P_t = \left[ \int_0^1 p_t(i)^{1-\xi} \, di \right]^{\frac{1}{1-\xi}}$$

and that the demand for variety $i$ is given by

$$\frac{c_t(i)}{C_t} = \left( \frac{p_t(i)}{P_t} \right)^{-\xi}.$$
b) Next, derive the FOCs with respect to $C_t$, $B_t$, and $N_t$.

A continuum of firms, indexed by $i$, produce differentiated varieties. Firm $i$ produces variety $i$ with a production function that is linear in labor $n_t(i)$ and is given by

$$y_t(i) = n_t(i)$$

where $y_t(i)$ is the output of firm $i$.

In this model, firms adjust prices every period and hire labor in a common, competitive market.

c) First, consider the cost minimization problem of firm $i$

$$\min \ W_t n_t(i)$$

subject to the production function

$$y_t(i) = n_t(i).$$

Derive the FOC with respect to $n_t(i)$. Use the FOC to derive an expression for the nominal marginal cost that is denoted by $\varphi_t$.

d) Next, consider the firm’s price-setting problem

$$\max \ \Gamma_t(i) = [p_t(i) - \varphi_t] \ c_t(i)$$

subject to the demand curve

$$\frac{c_t(i)}{C_t} = \left( \frac{p_t(i)}{P_t} \right)^{-\epsilon}.$$  

Derive the FOC with respect to $p_t(i)$.

e) Consider a symmetric equilibrium where all firms adjusting prices choose the same price, hire the same amount of labor, and produce the same amount of output. What is the relationship between $p_t(i)$ and $P_t$? How is $\varphi_t$ related to $P_t$?

In equilibrium, goods, labor hours, and bond markets clear.

f) Write down the market clearing conditions in terms of aggregate variables. Write down the aggregate production function. (Note that we are focusing on a symmetric equilibrium and that from our notation, $\Gamma_t = \int \Gamma_t(i)di$.)

g) Write down all the equilibrium conditions in terms of aggregate variables, including the market clearing conditions and the production function.

Then, answer the following questions by providing detailed explanations (Hint: You do not necessarily have to log-linearize and/or solve the model completely):

h) Does the shock $\psi_t$ affect aggregate output in this economy?

i) Does the shock $\psi_t$ affect the real interest rate in this economy?
3. Monetary model with price-adjustment cost and investment (50 points)

Consider a general equilibrium monetary model with investment where firms face price-adjustment costs.

**Consumer**

The representative consumer’s period utility is given by

\[ u(C_t, N_t) = \log C_t - \frac{N_t^{1+\phi}}{1 + \phi} \]

where \( C_t \) is consumption of the composite final good, \( N_t \) is hours supplied, and \( \phi \) is the inverse of the Prisch elasticity of labor supply. The consumer discounts the future by the discount factor \( \beta \) and maximizes expected discounted utility over the infinite horizon.

The representative consumer owns the capital stock and makes investment decisions. Capital depreciates at the rate \( \delta \). The evolution of aggregate capital \( K_t \) is given by

\[ K_{t+1} = I_t + (1 - \delta) K_t \]

where \( I_t \) is investment and \( 0 < \delta < 1 \) is the rate of depreciation. The consumer rents capital to firms in a competitive rental market. Note that the same composite final good is used for both consumption and investment.

The consumer can also save in one-period risk-less nominal bonds that are in zero net supply. Assume that the agent is subject to a no-Ponzi game constraint. The representative consumer owns all the firms in the economy and takes profit flow from them as given. The labor market is competitive.

**Firms**

There are two types of firms in the economy: composite final good producers and differentiated varieties producers.

**Composite final good producers**

This sector is perfectly competitive in which final good producers combine a continuum of differentiated varieties, indexed by \( i \), using a Constant Elasticity of Substitution technology

\[ Y_t = \left[ \int_0^1 y_t(i)^{\frac{1}{\varepsilon - 1}} \, di \right]^{\frac{\varepsilon - 1}{\varepsilon}} \]

where \( Y_t \) is aggregate output of the composite final good, \( y_t(i) \) is output of variety \( i \), and \( \varepsilon > 1 \) is the elasticity of substitution among the varieties.

**Differentiated varieties producers**

This sector is monopolistically competitive and firms face price adjustment costs. A continuum of firms, indexed by \( i \), produce the differentiated varieties. Firm \( i \) produces variety \( i \) with a Cobb-Douglas production function that uses labor \( (n_t(i)) \) and capital \( (k_t(i)) \) as inputs

\[ y_t(i) = A_t n_t(i)^{\alpha} k_t(i)^{1-\alpha} \]
where $y_t(i)$ is the output of firm $i$, $\alpha$ is the labor share, and $A_t$ is the aggregate TFP shock that follows an exogenous stationary AR(1) process. Firms rent capital and hire labor from common, competitive markets.

Changing nominal prices of differentiated varieties is costly and all firms face a price-adjustment cost $q(.)$ given by

$$q\left(\frac{p_t(i)}{p_{t-1}(i)}\right)$$

where $p_t(i)$ is the nominal price of variety $i$ and $q(.)$ is a convex function that satisfies $q(1) = q'(1) = 0$.

Firms face a downward sloping demand curve for their varieties and maximize expected discounted profits over the infinite horizon.

**Government**

There is no government spending in this model. The central bank conducts monetary policy using a feedback interest rate rule: it adjusts the nominal interest rate in response to (aggregate) inflation.

a) Write down all the conditions that characterize the (aggregate) equilibrium of this model. Note that for firms, we will focus on a symmetric equilibrium where all firms choose the same optimal price, produce the same output, and hire the same amount of labor and capital. Moreover, assume that the interest rate rule followed by the central bank satisfies the Taylor Principle and leads to a determinate equilibrium.

(Hint: There is a single aggregate price index in the model that you can derive given the constant elasticity of substitution technology of combining differentiated varieties. In deriving the aggregate resource constraint, consider carefully the role of the price-adjustment cost $q(.)$. For the firms’ problem, consider its cost-minimization problem first before writing down its optimal price-setting problem. Finally, if you feel not all necessary information is provided, carefully state any additional assumptions you need to make and proceed).

b) Log-linearize the (aggregate) equilibrium conditions from a) around a zero-(net) inflation, flexible-price, non-stochastic, steady-state.