Answer each question, they are worth 45 points a piece.
1. Optimal Taxation of Many Goods

Consider a static economy with a continuum of goods in which labor is the only factor of production. Utility is over each good plus leisure, and is given by

\[ \int_0^1 (\log c_i) di + \ell. \]

Production of each good is linear in the labor input with \( i \)-specific productivity, \( z_i \):

\[ y_i = z_i n_i. \]

All markets are competitive, the price of a good \( i \) is denoted \( p_i \), and all firms hire labor in the same market at wage \( w \).

1. Fully solve for a competitive equilibrium allocation and prices assuming that \( \ell \) is interior (i.e. ignore any upper bound on hours worked so that the math is clean).

2. Now suppose that the government must raise at least \( G \) worth of taxes by setting expenditure tax rates on each good, \( \tau_i \), and labor tax \( \tau_n \). All taxes are imposed on households, not firms. That is, \( G \leq \int_0^1 \tau_i p_i^* c_i^* di + \tau_n w^* n^* \). Show that the optimal tax rates are such that \( \tau_i = \tau_c \) for all \( i \in [0, 1] \).

3. Now consider a dynamic version of this model with one more good, so that the lifetime utility of a household is \( \sum_{t=0}^{\infty} \rho^t \left[ \log C_t + \int_0^1 \log c_{i,t} di + \ell_t \right] \). This new good uses both labor and capital to produce, so that \( Y_t = K_t^\alpha N_t^{1-\alpha} \), and capital accumulates as \( K_{t+1} = (1 - \delta) K_t + X_t \), where \( X_t \) is in units of \( Y_t \). Denote the rental rate of capital as \( r_t \), the tax on capital income as \( \tau_k,t \), and assume that the government must raise enough revenue in present value terms so that:

\[ \sum_{t=0}^{\infty} \rho^t \left[ \tau_{C,t} P_t C_t + \int_0^1 \tau_{c,t} p_{i,t} c_{i,t} di + \tau_{n,t} w_t n_t \right] \geq \sum_{t=0}^{\infty} \rho^t \left[ P_t G_t + \int_0^1 p_{i,t} g_{i,t} di \right] \]

Where government purchases are taken as given and \( \rho_t \) is the Arrow price that measures one unit of expenditures in period \( t \) in units of expenditures at period 0. Fully define a TDCE for this economy.

4. Is it true that \( \tau_{k,t} \to 0 \) in a steady state? Is \( \tau_{k,t} = 0 \) for \( t \geq 1 \)? Is it true that \( \tau_{i,t} \) is independent of \( i \)? Is \( \tau_{i,t} \) also independent of time? Explain your answers.
2. A Search Model With Realistic Benefits

This problem extends the McCall Search Model to make an unemployed person’s benefits depend on her past wage. That is, an unemployed worker gets flow utility of $b\tilde{w}$ in each period of being unemployed, where $\tilde{w}$ is the wage she earned at her previous job. While unemployed, she draws a job offer from a distribution on $[w, \bar{w}]$ with CDF $F(w)$, which she can either accept (and start working at in the next period) or reject (and continue as unemployed).

An employed worker receives wage $w$ in each period of employment and, at the end of each period, may be fired with probability $\sigma$. If she is fired, then she begins the next period as unemployed and her benefits are determined by $\tilde{w} = w$ until she accepts a new job.

1. Write the employed and unemployed problems as operators from a space of continuous bounded functions (with sup norm) into itself.

2. Prove that there is a unique fixed point to this operator.

3. Show that the unemployed follow a reservation wage strategy.

4. Is it true that the reservation wage is increasing in $\tilde{w}$? Prove your answer.
3. RBC model with distortionary labor taxation (45 points)

Consider a RBC style model with government spending where the government does not have access to lump-sum taxes.

Preferences

The representative household’s expected discounted utility is given by

\[ E_0 \sum_{t=0}^{\infty} \beta^t u(C_t, L_t) \]

where \( C_t \) is consumption, \( L_t \) is leisure, and \( 0 < \beta < 1 \) is the discount factor. The period utility \( u(\cdot) \) is strictly increasing, concave, and twice continuously differentiable.

Production technology

Output \((Y_t)\) is produced using a production function \( Y_t = A_t F(K_t, N_t) \)

where \( K_t \) is (pre-determined) capital, \( N_t \) is labor, and \( A_t \) is an aggregate random productivity shock that follows

\[ \log A_t = \rho_A \log A_{t-1} + \varepsilon_{A,t} \]

where \( \rho_A \in (0, 1) \) and \( \varepsilon_{A,t} \) is iid with finite mean and variance. The production function \( F(\cdot) \) is twice continuously differentiable, concave, and homogenous of degree one. \( F(\cdot) \) also satisfies the standard limiting conditions (the Inada conditions).

Accumulation technology

The evolution of capital is given by

\[ K_{t+1} = I_t + (1 - \delta) K_t \]

where \( I_t \) is investment, \( 0 < \delta < 1 \) is the rate of depreciation, and \( K_0 > 0 \) is given.

Government

The government finances exogenous government spending \( G \) by levying taxes on labor income and runs a balanced budget every period. Moreover, the initial stock of government debt is zero. Let \( \tau_t \) be the labor tax rate and \( w_t \) the pre-tax real wage earned by the household.

(Note that government spending is non-time varying and exogenous and does not enter household utility.)

Resource constraints

The total amount of time that the household has can be split into work and leisure. Normalizing the total amount of time each period to be 1, the time constraint is

\[ N_t + L_t = 1. \]

Moreover, since total output produced can be either consumed by the household and government or invested, another resource constraint is

\[ Y_t = C_t + I_t + G. \]

(a) Formulate, and carefully define, a price-taking version of the model above where the representative firm owns the capital stock and issues equity to finance its investment.
(b) Provide all conditions that characterize the equilibrium of the model you have formulated above.

(c) Assume the following functional forms for preferences and technology

\[ u(C_t, L_t) = \log C_t + \theta \log L_t \]

\[ F(K_t, N_t) = K_t^{1-\alpha} N_t^\alpha \]

where \( \theta > 0 \) and \( \alpha > 0 \) are model parameters. Given these assumptions, find the non-stochastic (that is, where \( \varepsilon_{A,t} = 0 \)) steady-state equilibrium as a function of the model parameters.

(d) Next, consider the following variation of the environment. Assume that there is a positive initial stock of one-period non-state contingent real government debt, \( B_0 > 0 \). Let the interest rate on government debt be \( r_t \). The government still runs a balanced budget every period. The rest of the environment is the same as before.

Provide all conditions that characterize the competitive equilibrium of this version of the model.
4. Optimal monetary policy in a sticky price model (45 points)
The central bank’s objective is to minimize the loss function

\[ \frac{1}{2} E_t \sum_{j=0}^{\infty} \beta^j \left[ \pi_{t+j}^2 + \phi_{x} \bar{x}_{t+j}^2 + \phi_{\pi} (\pi_{t+j} - \pi_{t-1+j})^2 \right] \]

subject to

\[ \pi_t = \beta E_t \pi_{t+1} - \alpha \pi_{t-1} + \kappa x_t + \varepsilon_{\pi,t} \]
\[ x_t = E_t x_{t+1} - (i_t - E_t \pi_{t+1}) + \varepsilon_{x,t} \]

where \( E_t \) is the conditional expectation operator, \( i_t \) is the central bank’s instrument, \( \pi_t \) and \( x_t \) are other endogenous model variables, and \( 0 < \beta < 1, \alpha > 0, \kappa > 0, \phi_{\pi} > 0, \) and \( \phi_x > 0 \) are model parameters. The central bank takes actions after the shocks \( \varepsilon_{\pi,t} \) and \( \varepsilon_{x,t} \) are realized. The shocks follow

\[ \varepsilon_{\pi,t} = \rho \varepsilon_{\pi,t-1} + v_{\pi,t} \]
\[ \varepsilon_{x,t} = \rho \varepsilon_{x,t-1} + v_{x,t} \]

where \( \rho \in (0,1) \) and \( v_{\pi,t}, v_{x,t} \) are iid with finite mean and variance.

(a) Suppose that the central bank can credibly commit at date \( t \) to a contingent path for \( i_{t+j} \).
Characterize, as far as you can, the solution to the optimal monetary policy problem above with commitment.

(b) Does your solution in (a) feature dynamic time inconsistency? Defend your answer.